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m-n COMPACT SPACES

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 U.N.B. DISSANAYAKE

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "m-n COMPACT SPACES" submitted by Upali Dissanayake in partial fulfilment of the requirements for the degree of Master of Science.

DEDICATION

To my conscientious teachers in Sri Lanka and Canada.

ABSTRACT

Let m and n be infinite cardinals, then a space X is said to be m-n compact if and only if every open cover of cardinality m has a sub-cover of cardinality strictly less than n. This concept covers the main three compactness-like properties, namely compactness, countable compactness and Lindelöf property.

In Chapter I, we study basic concepts about higher cardinals, product spaces and compact spaces. In Chapter II, we study m-n compact spaces and their product properties in detail by obtaining generalizations of some known theorems about products of m-n compact spaces. We also study weak topological sums of n-compact spaces and we prove that arbitrary products of Lindelöf, T_3, P-spaces are Lindelöf, provided the product is paracompact.

Our main tools are filters but in Chapter III we study Maximal-filters and some of their applications to compact-like spaces in a non-detailed manner. This chapter is mainly to give an initiative step to beging the work on Filter Techniques and Compactness-like Properties. We also give an introduction to weakly m-n compact spaces since these spaces are generalizations of m-n compact spaces.

PREFACE

This thesis is designed to study m-n Compact Spaces using Generalized filters. The material of this thesis is divided into three chapters and our main goal is to study productivity of compactness-like properties in a general setting.

In Chapter I we include basic concepts of cardinal arithmetic, product spaces and compact spaces. In Chapter II we study m-n compact spaces and their product properties in detail and in Chapter III we study some applications of maximal filters to compact-like spaces. The concept of m-n compactness and its generalizations have been studied by J.E. Vaughan, W.W. Comfort, N. Noble, M. Ulmer and others.

Chapter I is devoted to basic properties of infinite cardinals, generalized products, compact spaces and the Stone-Čech compactification. In the study of cardinals we pay more attention to regular and singular properties; also we rely on the filter description of the Stone-Čech compactification and we denote by βX. In the preceding chapters points of βX and the corresponding ultrafilters are used interchangeably.

In Chapter II we mainly study product theorems about compact-like spaces in a general setting and we extend some theorems in partial generality. The properties which are stronger than m-n compactness have been studied by J.E. Vaughan and others. The property $1_{m,n}$ is in this nature and it has interesting applications to products of compact-like spaces. We study this property and its applications in the Sections 3.1 - 3.4.

In Chapter III we study some properties of maximum filters and we use those properties to study a hard example about strongly α-compact

spaces. The concept of Γ-compactness is defined in terms of ultrafilters and it is stronger than countable compactness. The work in this nature can be considered as applications of maximal filters to compactness-like properties.

The symbols m and n in this thesis denote infinite cardinals unless otherwise stated and undefined terminology follows that of Willard (23). The main spaces and main key words are listed for quick reference and for the convenience of the reader.

We also give a reference list of examples which are related to the main spaces of this thesis.

Finally we wish to mention that weakly m-n compact spaces are generalizations of m-n compact spaces and this concept also covers varieties of compact-like spaces such as H(i)-spaces, feebly compact spaces, weakly-Lindelof spaces for suitable m and n.

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CONTENTS

MAIN SPACES

Let m and n be infinite cardinals. Then we have the following:

Bicompact spaces (olden days - Alexandroff and Uryshon)

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Compact spaces

Γ-compact spaces (new concept -1970)

Strongly m-n compact spaces

(n = regular and $m^{\underline{n}} = m$)

The property $1_{m,n}$

countably compact spaces (Fechet-metric spaces)

m-n compact spaces

Lindelöf spaces (Sierprinski - 1921)

H(i)-spaces

(Scarborough and Stone)

Weakly m-n compact spaces

weakly-countably compact spaces

weakly-Lindelöf Spaces (present day interest)

Main Key Words / Symbols

I. 1. Regular and singular cardinals

2. Strongly γ-inaccessible cardinals

3. Generalized products

4. γ-weak topological sums

5. Density character of a space

6. Dictionary order relation

7. Compact spaces

8. Stone-Cêch compactification

9. Filters, Filter bases

10. Ultrafilters (Maximal Filters)

II. 11. Character of a space

12. $<n$-discrete spaces, p-spaces

13. Zero sets

14. m-n compact spaces

15. m-n filters (m-n filter bases)

16. Strongly m-n compact spaces

17. The property $1_{m,n}$

18. α-bounded spaces

III. 19. Discrete spaces

20. Principal and non-principal untrafilters

21. k-Uniform ultrafilters

22. Measurable cardinals

23. Γ-compact spaces , Γ-limit points

24. Weakly m-n compact spaces

1. $|X|$-cardinality of X

2. $|X|^+$-First cardinal strictly larger than $|X|$.

3. 2^α-$\exp(\alpha)$.

4. (GCH)-Generalized continuum hypothesis is assumed without explicit mention.

5. $P(X)$-Power set of X .

6. Φ-empty set

7. $\bigcup\mathcal{F}$ - $\bigcup\{F: F \in \mathcal{F}\}$, $\mathcal{F} \subset P(X)$.

8. $\bigcap\mathcal{F}$ - $\bigcap\{F: F \in \mathcal{F}\}$

9. G^o-interior of $G = \mathrm{Int}_X G$, $\overline{F}$ -closure of $F = C\ell_X F$.

10. $\bigcap\overline{\mathcal{F}}$-$\bigcap\{\bar{F}: F \in \mathcal{F}\}$.

11. $\mathcal{V}_x$-neighbourhood system at $x \in X$.

12. $\mathcal{V}_A$-neighbourhood system at $A \in P(X)$.

13. $C(X)$-set of all continuous real-valued functions on X .

14. $C^*(X)$-set of all continuous real-valued bounded functions.

15. $\mathcal{F}/S$-$\{F \cap S: F \in \mathcal{F}\}$.

16. $\mathcal{F} \vee \mathcal{G}$-collection of all finite intersection of members of $\mathcal{F}$ and $\mathcal{G}$; $\mathcal{F}, \mathcal{G} \subset P(X)$.

17. $ad_X\mathcal{G}$-$\bigcap\overline{\mathcal{G}}$.

18. $f(\mathcal{F})$-$\{f(F): F \in \mathcal{F}\}$.

19. $f^{-1}(\mathcal{F})$-$\{f(F): F \in \mathcal{F}\}$.

20. Space X-Topological space X.

21. $\mathbb{N}$-Set of all positive integers of X .

22. Q-set of all rational numbers.

23. $\mathbb{R}$-set of all real numbers

24. $\gamma(\prod_{i \in I} X_i)$-$\gamma$-weak topological sums of the spaces $\{X_i: i \in I\}$.

CHAPTER I

PRELIMINARIES

1. Higher Cardinals

In this section we study some properties of higher cardinals which will be useful in the latter chapters. We assume elementary facts about addition, multiplication and exponentiation of cardinal numbers and we shall state standard theorems about cardinal numbers without proofs [which can be found in H.B. ENDERTON - (6)].

1.1. Cardinal Arithemetic.

A. THEOREM. Let k and λ be any two cardinal numbers such that the larger is infinite and the smaller is non-zero. Then

$$k + \lambda = k \cdot \lambda = \max\cdot(k,\lambda).$$

This is called the ABSORPTION LAW of Cardinal Arithmetic.

B. THEOREM. Let $\mathscr{A}$ be a collection of sets. Suppose each member of $\mathscr{A}$ has cardinality less than or equal to k, then $|\bigcup \mathscr{A}| \leq k\cdot|\mathscr{A}|$ where $|\mathscr{A}|$ denotes the cardinality of $\mathscr{A}$. This is a generalization of the result that the countable union of countable sets is countable.

C. Notation. $\alpha^{\overline{k}} = \sum \{\alpha^{\gamma}: \gamma < k\}$ where α,γ and k are cardinal numbers.

Example. Let α be an infinite cardinal number. Then $\alpha^{\aleph_0} = \alpha$.

D. LEMMA. Let X be an infinite set and $\mathcal{A}$ be the set of all finite subsets of X. Then the cardinality of $\mathcal{A}$ is $|X|$.

Proof. Let $\mathcal{A}_n = \{A \in P(X) : |A| = n\}$ for $n = 1,2,\cdots\cdots$. Then $|\mathcal{A}_n| \le |X^n| = |X|$. We note that $\mathcal{A} = \bigcup \mathcal{A}_n$ and hence $|\mathcal{A}| \le \aleph_0 \cdot |X| = |X|$. It is easy to see that $|\mathcal{A}| \ge |X|$. Hence $|\mathcal{A}| = |X|$.

1.2. Regular and Singular Cardinals.

A. Definition. Let α be an infinite cardinal. Then the smallest cardinal β for which there is a family $\{\alpha_\xi : \xi < \beta\}$ of cardinals such that $\alpha_\xi < \alpha$ for $\xi < \beta$ and $\sum_{\xi<\beta} \alpha_\xi = \alpha$ is called the cofinality of α and denoted by $Cf(\alpha)$.

Let G be a set with cardinality α. Suppose S is a subset of G such that $\text{Sup}.\ S = \alpha$. Then S is said to be cofinal in G. By the above definition cofinality of G = minimum $\{|S| : S \text{ is cofinal in } G\}$.

Note. G itself is cofinal in G. Hence for any cardinal α, $\alpha \ge Cf(\alpha)$.

B. Definition. A cardinal α is said to be regular if and only if $Cf(\alpha) = \alpha$ and otherwise α is said to be a singular cardinal.

Example. Consider $\aleph_0, \aleph_1, \cdots\cdots, \aleph_\omega$ We know that $\aleph_\omega = \text{Sup}.\ \{\aleph_0, \aleph_1 \cdots\cdots\}$. It is easy to see that $Cf(\aleph_0) = \aleph_0$ and $Cf(\aleph_\omega) = \aleph_0$. Therefore $\aleph_0$ is a regular cardinal and $\aleph_\omega$ is a singular cardinal.

Note. Let α be a regular cardinal. Let $\{\alpha_\mu : \mu \in I\}$ be a family of cardinals such that $\alpha_\mu < \alpha$ for all $\mu \in I$. Suppose $|I| < \alpha$. Then $\sum_{\mu \in I} \alpha_\mu < \alpha$.

LEMMA. (GCH) Let $n \leq m$ and m be regular, then $m^{\bar{n}} = m$.

Proof. We note that $m^\alpha = m$ for $\alpha < n$ and hence we have $m^{\bar{n}} = \sum \{ m^\alpha : \alpha < n\} = m$.

C. THEOREM. For any infinite cardinal α we have the following:

(i) $Cf(\alpha)$ is regular,

(ii) $\alpha^{\bar{Cf(\alpha)}} > \alpha$.

Proof. See [6] - 9S.

Example. Let $\alpha = \aleph_\omega$. Then $Cf(\alpha) = \aleph_0$. Hence $\aleph_\omega^{\aleph_0} > \aleph_\omega$ but $\aleph_\omega^{\bar{\aleph}_0} = \aleph_\omega$.

D. THEOREM. Let α be an infinite cardinal. Then the cofinality of α is the least cardinal number β such that α can be decomposed into the union of β sets, each having cardinality less than α.

Proof. See [Enderton - (6), 9T].

Note. $Cf(\aleph_1) = \aleph_1$.

By Cantor's Theorem we know that for any cardinal α $\exp(\alpha) > \alpha$. This result has improved by KÖNIG; we shall state that as a theorem.

E. THEOREM. Let α be an infinite cardinal, then the cofinality of $(\exp \alpha) > \alpha$.

Examples. (i) Cofinality of $\aleph_\omega = \aleph_o$ and cofinality of $(\exp \aleph_0) > \aleph_0$ (By KÖNIG'S Theorem). Therefore $\exp \aleph_o \neq \aleph_\omega$.

(ii) Let α be an infinite cardinal, then $\alpha^\alpha = 2^\alpha$.

(iii) The number of continuous functions from $\mathbb{R} \to \mathbb{R}$ is $|C(\mathbb{R})| = 2^{\aleph_o}$.

(iv) Let α be infinite and $2 \leq k \leq \alpha$, then $k^\alpha = 2^\alpha$.

1.3. Inaccessible Cardinals.

A. Definition. A cardinal α is said to be strongly inaccessible if and only it satisfies the following conditions:

(i) $\alpha > \aleph_o$,

(ii) $2^\beta < \alpha$ for all $\beta < \alpha$,

(iii) α is regular.

Note: A cardinal which satisfies (ii) is called a strong-limit cardinal.

Example. Consider $\aleph_o, \aleph_1, \cdots, \aleph_\omega$. By GCH we have $2^{\aleph_n} = \aleph_{n+1}$. Therefore we have $\aleph_\omega > 2^{\aleph_n}$ for all $n = 0,1, \cdots$. Hence $\aleph_\omega$ is a strong-limit cardinal.

B. Definition. A cardinal which satisfies (ii) and (iii) of (A) is called an inaccessible cardinal.

C. Definition. Let α and γ be cardinals, such that $\gamma \leq \alpha$. Then α is said to be strongly γ-inaccessible if $\beta^k < \alpha$ for $\beta < \alpha$ and $k < \gamma$.

THEOREM. Let α be a regular cardinal with $\alpha > \gamma \geq \aleph_o$. Then

the following are equivalent: [see - (4)]

(i) α is strongly γ-inaccessible,

(ii) If $k < \gamma$ and $\alpha_\mu < \alpha$ for all $\mu < k$, then $\prod_{\mu<k} \alpha_\mu < \alpha$,

(iii) $\beta^{\underline{\gamma}} < \alpha$ for all $\beta < \alpha$.

D. Definition. If a limit cardinal is regular, then it is called a weakly inaccessible cardinal.

The existence of these type of cardinals is not known.

Let n and γ be cardinals, such that $n \geq \gamma$. Let $\overline{\gamma} = \gamma$ (γ = regular), $\overline{\gamma} = \gamma^+$ (γ = singular). Suppose n is regular and strongly γ-inaccessible, then we have $\overline{\gamma} \leq n$.

For. We note that if γ is singular, then since n is regular and strongly γ-inaccessible, $\gamma^{Cf(\gamma)} < n$. We also know that by Theorem 1.2-C, $\gamma^{Cf(\gamma)} > \gamma$ and hence we have $n \geq \gamma^+$.

1.4. Large Cardinals.

A. Definitions. (i) A $\{0,1\}$-valued measure on X is a countably additive function defined on the power set of X, $P(X)$ assuming only the values $0,1$.

(ii) If every non-zero, $\{0,1\}$-valued measure assigns measure one to some one-element set of X, then $|X|$ is said to be non-measurable.

We can show that there is a one to one correspondence between set of all Maximal Filters on X and non-zero finitely additive $\{0,1\}$-valued set functions defined on X.

Let $\mathcal{U}$ be a maximal filter on X and $X_{\mathcal{U}}$ be the corresponding finitely additive set function. If $\mathcal{U}$ has countable intersection property,

then $X_{\mathcal{U}}$ is a measure on X.

B. Definition. Using the notation in the section (A), $X_{\mathcal{U}}$ is free or fixed according as $\mathcal{U}$ is free or fixed.

THEOREM. Let X be a set with cardinality α. Then α is non-measurable if every $\{0,1\}$-valued measure $X_{\mathcal{U}}$ on X is fixed.

Note. Finding non-measurable cardinals is purely set-theoretic.

C. Some Properties of Non-Measurable Cardinals.

(i) Every cardinal smaller than a non-measurable cardinal is non-measurable.

(ii) Every non-measurable sum of non-measurable cardinals is non-measurable.

(iii) If m is non-measurable, then $\exp(m)$ is non-measurable.

By the definition $\aleph_0$ is a non-measurable cardinal and therefore the class of all non-measurable cardinals is very extensive. The existence of a measurable cardinal is not known, but this will not prevent us from studying these large cardinals.

D. Definition. A cardinal α is said to be measurable if there exists a $\{0,1\}$-valued measure μ on a set X of cardinality α and satisfying the following conditions:

(i) $\mu(\{x\}) = 0 \quad \forall \; x \in X$,

(ii) $\mu(X) = 1$,

(iii) If $\{X_i : i \in I\}$ is a family of disjoint subsets of X with $|I| < \alpha$, then $\mu\left(\bigcup_{i\in I} X_i\right) = \sum_{i\in I} \mu X_i$.

The property (iii) is called <u>$< \alpha$-additivity</u> of the measure μ. The non-triviality of μ follows from (ii). It is easy to see that the sets of measure 1 form a maximal filter on X which is closed under $< \alpha$-intersections. Conversely each free (non-principal) maximal filter on X which is closed under $< \alpha$-intersections defines a measure μ with the following properties:

(i) μ is $\{0,1\}$-valued,

(ii) $\mu(\{x\}) = 0 \ \forall\, x \in X$,

(iii) $\mu(x) = 1$,

(iv) μ is $< \alpha$-additive.

We shall prove that every measurable cardinal is strongly inaccessible.

<u>THEOREM.</u> Let α be a measurable cardinal. Then α satisfies the following properties:

(i) $\alpha > \aleph_o$,

(ii) α is regular,

(iii) α is a strong-limit cardinal.

<u>Proof.</u> (i) $\aleph_o$ is non-measurable and therefore $\alpha > \aleph_o$.

(ii) Let $|X| = \alpha$. Then there exists a decomposition $\{X_\gamma : \gamma \in I\}$ of X where $|I| = Cf(\alpha)$ and $|X_\gamma| < \alpha$. Since α is measurable there exists a $\{0,1\}$-valued, non-zero, $< \alpha$-additive measure μ on X. If α is singular, then $\mu(X) = 0$. This is a contradiction. Hence α is regular.

(iii) Suppose $\beta < \alpha \leq 2^\beta$. Then we can assume that $X \subseteq \{\{0,1\}\text{-valued } \beta\text{-sequences}\}$. For each $\gamma < \beta$ define $i_\gamma \in \{0,1\}$ such that $\mu\{x \in X : x(\gamma) = i_\gamma\} = 1$. Let x_0 be defined by $x_0(\gamma) = i_\gamma$ for all $\gamma < \beta$. Let $x \in X$ and $X(\gamma) \neq i_\gamma$ for some $\gamma < \beta$. Then

$x \in X_\gamma = \{x \in X : x(\gamma) \neq i_\gamma\}$. We note that $\mu X_\gamma = 0$. Hence $\mu\left(\bigcup_{\gamma<\beta} X_\gamma\right) = 0$. This leads to a contradiction that μ is trivial. Hence α is a strong-limit cardinal.

Notes. 1. Let I be a non-empty set and let $\{\alpha_i\}, \{\beta_i\}$ a family of cardinals such that $\alpha_i < \beta_i$ for $i \in I$. Then $\sum_{i\in I} \alpha_i < \prod_{i\in I} \beta_i$.

This is called KÖNIG'S Theorem according to (3)-1.19 and by this theorem we note that $\alpha < \alpha^{Cf(\alpha)}$. ($\alpha = \sum_{\xi<Cf(\alpha)} \alpha_\xi$ where $\alpha_\xi < \alpha$).

2. Let I be a set with cardinality α and let $P_k(I) = \{S \in P(I) : |S| < k\}$. Suppose $\alpha^+ \geq k + \aleph_0$ then $|P_k(I)| = \alpha^{\overline{k}}$. [See (2) - 1.22.]

3. If $\alpha \geq 2$, then $\alpha^{\overline{k}} \geq k$. This follows from the fact that $\alpha^{\overline{k}} = \sum_{\xi<k} \alpha^\xi$ and $\alpha^\beta = \sum_{\xi\in\beta^+-\beta} \alpha^{|\xi|}$. We also note that $|\xi| = \beta$ for all $\xi \in \beta^+-\beta$ and we consider a cardinal α as a least ordinal with cardinality α.

4. Let α be a cardinal, then α^+ is the least cardinal β such that $\alpha < \beta$. Such cardinal exists, because $\alpha < 2^\alpha$ and the set of cardinals less than or equal to 2^α is well-ordered.

Let α be a cardinal and let $\alpha = \beta^+$ for some β, then α is non-limit and every infinite non-limit cardinal is regular because $Cf(2^\alpha) > \alpha$.

2. Product Spaces.

In this section we shall study basic properties of products of topological spaces in a more general setting. The main interest is on γ-weak topological sums of the factor spaces of a product space.

2.1. Elementary Facts.

A. Definition. The product topology on $X = \prod_{\alpha \in I} X_\alpha$ is the topology generated by basic open sets of the form $U = \prod_{\alpha \in I} U_\alpha$ where

(i) U_α is open in X_α for each $\alpha \in I$,

(ii) $U_\alpha = X_\alpha$ for all α except for finitely many.

B. Note. If $U_\alpha = X_\alpha$ for all $\alpha \in I$ except for finitely many $\alpha = \alpha_1, \alpha_2, \cdots\cdots \alpha_n$, then $\prod_{\alpha \in I} U_\alpha = \bigcap_{i=1}^{n} \Pi_{\alpha_i}^{-1}\left(U_{\alpha_i}\right)$. Hence $\{\Pi_\alpha^{-1}(U_\alpha) : \alpha \in I$ and U_α is open in $X_\alpha\}$ is a sub-base for the product topology. Further more the sets U_α can be restricted to come from some fixed sub-base in X_α.

C. Notation. Let $X = \prod_{i \in I} X_i$ be a product space with the product topology. For each non-empty set $I' \subset I$, we set $X_{I'} = \prod_{i \in I'} X_i$ and let $\Pi_{I'} : X \to X_{I'}$ be the projection map. In the special case $I' = \{\alpha\}$, $\Pi_{I'} \equiv \Pi_\alpha : X \to X_\alpha$ is the usual projection map.

D. Definition. Let $U = \prod_{i \in I} U_i$ where U_i is a subset of X_i for each i. The range of U is defined as $\mathcal{R}(U) = \{i \in I : U_i \neq X_i\}$.

Example. Let U be a basic open set in the product space $X = \prod_{i \in I} X_i$. Then $|\mathcal{R}(U)|$ is less than $\aleph_0$.

Note. Let $U = \prod_{i \in I} U_i$ where U_i is a subset of X_i for each i. Then we have the following easy results:

(i) $|\mathcal{R}(\Pi_{I'}(U))| \leq |\mathcal{R}(U)|$,

(ii) $|\mathcal{R}(\Pi^{-1}_{I'}(U_{I'}))| = |\mathcal{R}(U_{I'})|$

(iii) If $V \supset U$, then $\Pi_i(V) = X_i$ for all $i \in I - \mathcal{R}(U)$.

LEMMA. Let $U = \prod_{i\in I} U_i$ and

$V = \prod_{i\in I} V_i$ where U_i, V_i are subsets of X_i for each i. Then the following are equivalent:

(i) $U \cap V = \Phi$,

(ii) $U_i \cap V_i = \Phi$ for some $i \in \mathcal{R}(U) \cap \mathcal{R}(V)$,

(iii) $\mathcal{R}(U) \cap \mathcal{R}(V) \neq \Phi$ and $\Pi_{I'}(U) \cap \Pi_{I'}(V) = \Phi$ where $I \supseteq I' \supseteq \mathcal{R}(U) \cap \mathcal{R}(V)$.

Proof. (i) $\Rightarrow$ (ii): Trivial,

(ii) $\Rightarrow$ (iii): Trivial,

(iii) $\Rightarrow$ (i): Let $i \in \mathcal{R}(U) \cap \mathcal{R}(V)$ and take $I' = \{i\}$.

2.2. Generalized Products.

A. Definition. The topology generated by the basic open sets of the form $U = \prod_{i\in I} U_i$ where U_i is open in X_i and $|\mathcal{R}(U)| < \alpha$ is called the α-box topology on the product $X = \prod_{i\in I} X_i$ and is denoted by $\left(\prod_{i\in I} X_i\right)_\alpha$ where $\alpha \geq \aleph_o$.

B. Special Cases. Let $\alpha = \aleph_o$ then the α-box topology is the usual product topology on the product $X = \prod_{i\in I} X_i$. Let $\alpha = |I|^+$. Then the α-box topology is the box topology on the product $X = \prod_{i\in I} X_i$. This topology has large number of open sets compared to the product topology.

C. LEMMA. Let $\Pi_{I'} : \left(\prod_{i\in I} X_i\right)_\alpha \to \left(\prod_{i\in I'} X_i\right)_\alpha$. Then we have the following:

(i) $\Pi_{I'}$ is onto (assume the axiom choice),

(ii) $\Pi_{I'}$ is continuous ,

(iii) $\Pi_{I'}$ is open.

Proof. Follows from the definition 2.2-A and the note 2.1-D of this chapter.

Note. Let $I' = \{i\}$. Then Π_i is continuous, open and onto with respect to the α-Box topology on the product space $X = \prod_{i\in I} X_i$.

D. LEMMA. Let $X(I') = \left\{x \in X = \prod_{i\in I} X_i : x_i = a_i \text{ for } i \in I-I'\right\}$. Then $(X(I'))_\alpha$ is homeomorphic to $(X_{I'})_\alpha$.

Proof. We note that $(X(I'))_\alpha$ is a subspace of $\left(\prod_{i\in I} X_i\right)_\alpha$ and $\Pi_{I'}/X(I')$ is one to one. Hence by lemma 2.2-C of Chapter I, $(X(I'))_\alpha$ is homeomorphic to $(X_{I'})_\alpha$ under the homeomorphism $\Pi_{I'}/X(I')$.

2.3. γ-Weak Topological Sums.

A. Definition. Let $X = \prod_{i\in I} X_i$ and 'a' be a fixed point in X. Then we define the γ-weak topological sum of $\{X_i : i \in I\}$ as follows:

$$\gamma\left(\prod_{i\in I} X_i\right) = \{x \in X : |\{i \in I : x_i \neq a_i\}| < \gamma\}$$

B. Note. Let $I_{\underline{\gamma}} = \{I' \subset I : |I'| < \gamma\}$ and $X(I') = \{x \in X : x_i = a_i$ for all $i \in I-I'\}$. Then $\gamma\left(\prod_{i\in I} X_i\right) = \bigcup \{X(I') : I' \in I_\gamma\}$.

C. <u>LEMMA.</u> Let a' be a fixed point in $X = \prod_{i\in I} X_i$. Then $\gamma\left(\prod_{i\in I} X_i\right)$ is dense in $\left(\prod_{i\in I} X_i\right)_\alpha$ where $\alpha \leq \gamma$.

<u>Proof.</u> Let $U = \prod_{i\in I} U_i$ be a basic open set in $\left(\prod_{i\in I} X_i\right)_\alpha$. We define the point $x \in X$ as follows:

$$x_i = a_i \text{ for all } i \in I - R(U) \text{ and } x_i \in U_i \text{ for}$$

$i \in R(U)$. Then clearly $x \in U$ and we shall prove that $x \in \gamma\left(\prod_{i\in I} X_i\right)$. We note that

$$\begin{aligned} |\{i \in I : x_i \neq a_i\}| &\leq |R(U)|, \\ &< \alpha, \\ &\leq \gamma. \end{aligned}$$

Hence $\underline{x} \in \gamma\left(\prod_{i\in I} X_i\right)$. Therefore $\gamma\left(\prod_{i\in I} X_i\right)$ is dense in $\left(\prod_{i\in I} X_i\right)_\alpha$.

D. <u>Note.</u> Let $\alpha = \gamma = \aleph_0$. Then by the previous lemma $\aleph_0\left(\prod_{i\in I} X_i\right)$ = weak topological sum of $\{X_i : i \in I\}$ is dense in $\prod_{i\in I} X_i$.

2.4. <u>Density Character.</u>

A. <u>Definition.</u> Let X be a topological space. Then the density character of X, denoted by $d(X)$, is the least cardinal which is equal to the cardinal number of a dense subset of X.

<u>Examples.</u> Let $\mathbb{R}$ = reals, X = discrete space, S = Indiscrete space, I = unit interval.

(i) $d(\mathbb{R}) = \aleph_0$,

(ii) $d(X) = |X|$,

(iii) $d(S) = 1$,

(iv) $d(I) = \aleph_0$.

B. <u>Definition.</u> Let A and B be two sets with order relations $<_A$ and $<_B$. We define an order relation $<$ on $A \times B$ as follows:

$$a_1 \times b_1 < a_2 \times b_2 \quad \text{if} \quad a_1 <_A a_2 \quad \text{or if} \quad a_1 = a_2 \quad \text{and} \quad b_1 <_B b_2.$$

This is called the <u>dictionary order</u> relation on $A \times B$.

<u>Example.</u> Let $H = I \times I$ with dictionary order topology. Then H has uncountable number of disjoint open sets. Hence any dense subset of H has atleast $2^{\aleph_0}$ elements but $|H| = 2^{\aleph_0}$. Therefore $d(H) = 2^{\aleph_0}$.

C. <u>LEMMA.</u> Let α be an infinite cardinal and $X = \prod_{i \in I} X_i$ with $|X_i| \geq 2$ for $i \in I$ and $|I| > 2^{\alpha}$. Then $d(X) > \alpha$.

<u>Proof.</u> Let U_i and V_i be disjoint non-empty open subsets of X_i for $i \in I$ and let $|D| = d(X)$. The function $h : I \to P(D)$ defined by $h(i) = \pi_i^{-1}(U_i) \cap D$ is one to one . <u>For.</u>

(If i and j are different elements of I and $x \in D \cap \pi_i^{-1}(U_i) \cap \pi_j^{-1}(V_j)$ then $x \in \pi_i^{-1}(U_i) \cap D$ and $x \notin \pi_j^{-1}(U_j) \cap D$).

Therefore $|P(D)| \geq |I| > 2^{\alpha}$ and hence $d(X) = |D| > \alpha$.

D. <u>Definition.</u> Let α be an infinite cardinal number. The logarithm of α denoted by $\log \alpha$, is the least cardinal β such that $\alpha \leq 2^{\beta}$.

<u>Example.</u> $\log (\aleph_0) = \aleph_0 = \log (\aleph_1)$.

<u>LEMMA.</u> Let $\alpha \geq \aleph_0$ and let $\{X_i : i \in I\}$ be a set of spaces such that $d(X_i) \leq \alpha$ for $i \in I$. If $|I| \leq 2^{\alpha}$, then $d\left(\prod_{i \in I} X_i\right) \leq \alpha$.

Proof. See [W.W. Comfort -(4)] - 2.2 .

THEOREM. Let $\{X_i : i \in I\}$ be a family of spaces such that $|X_i| \geq 2$ for $i \in I$ and $|I| \geq \aleph_0$. Then $d\left(\prod_{i\in I} X_i\right) = \max. \{\log |I|,$ $\sup \{d(X_i) : i \in I\}\}$.

Proof. Let $X = \prod_{i\in I} X_i$ and

$$\beta = \max. \{\log |I|, \text{Sup.} \{d(X_i) : i \in I\}\}.$$

Then $d(X_i) \leq \beta$ for $i \in I$ and $2^\beta \geq I$. By the previous lemma $d(X) \leq \beta$. Suppose $d(X) = |D|$. Let U_i be an open subset of X_i. Then $U_i \cap \pi_i(D) \neq \Phi$. Hence $d(X_i) \leq |\pi_i(D)| \leq |D| = d(X)$ for $i \in I$. We shall prove that $\log |I| \leq d(X)$. Suppose not, then $|I| > 2^{d(X)}$. Then by lemma (C) of this sub-section $d(X) > d(X)$.

Therefore $d(X) \geq \beta$. Hence $d(X) = \beta$.

3. Compact Spaces.

This section is devoted to a brief study of the notion of compactness using filters. We shall give some important lemmas and theorems in this section which are useful in the latter chapters.

3.1. Elementary Facts.

A. Definition. A space X is compact if and only if every open cover of X has a finite sub-cover and X is $H(i)$ if and only if every open cover of X has a finite sub-family whose closure cover X.

Trivially, every compact space is $H(i)$.

B. Example. (Particular point topology)

Let X be an infinite set. We shall define a topology on X as follows:

Let $\tau = \{G : d \in G\} \cup \{\Phi\}$ where $d \in X$. Then (X,τ) is a topological space and the topology τ is called the particular point topology on X.

It is easy to see that (X,τ) is $H(i)$ but not compact. Hence the property compactness is strictly stronger than $H(i)$.

C. THEOREM. Let X be a regular space. Then the following are equivalent:

(i) X is compact,

(ii) X is $H(i)$.

Proof. (i) $\Rightarrow$(ii) : Trivial.

(ii) $\Rightarrow$ (i): Let $\mathscr{G}$ be an open cover of X. Using the regularity of the space X, we can find an open cover $\mathscr{H}$ of X such that $\overline{\mathscr{H}}$ refines $\mathscr{G}$. Since X is $H(i)$, $\mathscr{H}$ has a finite sub-collection whose closures cover X. Hence $\mathscr{G}$ has a finite sub-cover.

THEOREM. The property compactness and $H(i)$ are preserved under continuous images.

Proof. Follows from the definition (A).

3.2. Filter Characterization.

A. Definition. A family $\mathscr{F}$ of subsets of X has the finite intersection property if and only if the intersection of any finite sub-collection from $\mathscr{F}$ is non-empty.

Families with the finite intersection property are called

Filter Sub-bases.

B. THEOREM. Let X be a topological space. Then the following are equivalent:

(i) X is compact,

(ii) Every family of closed subsets of X with the finite intersection property has a non-empty intersection,

(iii) Every filter in X has an adherent point in X,

(iv) Every Maximal Filter in X converges in X.

The above theorem is standard and well-known.

3.3. Compact Sets.

A. Definition. A subset E of X is compact if and only if every cover of E by open subsets of X has a finite sub-cover.

Note. E is compact in X if and only if E is compact with respect to its subspace topology.

B. Some Properties of Compact Sets.

1. Every closed subset of a compact space is compact.
2. A compact subset of a T_2 space is closed.
3. In a T_2 space disjoint compact subsets can be separated by disjoint open sets.
4. A compact set and disjoint closed set in a regular space can be separated by disjoint open sets.

C. THEOREM. Let $X = \prod_{i \in I} X_i$ and K_i be a compact subset of X_i for $i \in I$. Let V be an open neighbourhood of $K = \prod_{i \in I} K_i$. Then there

exist open neighbourhoods W_i of K_i such that $V \supset \prod_{i \in I} W_i$ where $W_i = X_i$ for all $i \in I$, except for finitely many.

Proof. Since V is an open set in $X = \prod_{i \in I} X_i$, there exists an $I' \subset I$ such that $\Pi_i(V) = X_i$ for all $i \in I - I'$ and I' is a finite set. Let $I' = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $x_{\alpha_i} \in K_{\alpha_i}$ for $i = 1,2,\cdots,n$. Then, since V_{α_i} is a neighbourhood of x_{α_i}, there exist open sets Q_{α_i} such that $x_{\alpha_i} \in Q_{\alpha_i}$ for $i = 1,2,\cdots,n$ and $\prod_{i=1}^{n} Q_{\alpha_i} \subset \prod_{i=1}^{n} V_{\alpha_i}$. Fix $x_{\alpha_i} \in K_{\alpha_i}$ for $i = 1,2,\cdots,n-1$ and vary $x_{\alpha_n} \in K_{\alpha_n}$. Since K_{α_n} is compact, there exists an open cover $\left\{\prod_{i=1}^{n} Q_{\alpha_i}^{(s)} : s = 1,2,\cdots,q\right\}$ of the product set $\{x_{\alpha_1}\} \times \cdots \times \{x_{\alpha_{n-1}}\} \times K_{\alpha_n}$. Let $Q_{\alpha_i}^{(n)} = \bigcap_{s=1}^{q} Q_{\alpha_i}^{(s)}$ for $i = 1,2,\cdots,n-1$ and $Q_{\alpha_n}^{(n)} = \bigcup_{s=1}^{q} Q_{\alpha_n}^{(s)}$. Then $\{x_{\alpha_1}\} \times \cdots \times \{x_{\alpha_{n-1}}\} \times K_{\alpha_n} \subseteq \prod_{i=1}^{n} Q_{\alpha_i}^{(n)} \subseteq \prod_{i=1}^{n} V_{\alpha_i}$. By repeating this process we get $\{W_{\alpha_i} : i = 1,2,\cdots n\}$ such that $\prod_{i=1}^{n} K_{\alpha_i} \subseteq \prod_{i=1}^{n} W_{\alpha_i} \subseteq \prod_{i=1}^{n} V_{\alpha_i}$. Hence $V \supseteq \prod_{i=1}^{n} W_i$ where $W_i = X_i$ for $i \in I - I'$. This completes the proof of the theorem.

Note. This is a trivial extension of Wallace's Theorem On the Product of Two Compact Sets.

D. LEMMA. Let K be a compact subset of X and $\mathcal{F}$ be a filter base on X. Suppose every neighbourhood of K contains a member of $\mathcal{F}$, then $\cap\overline{\mathcal{F}}$ will contain a point of K.

Proof. Suppose $\cap\overline{\mathcal{F}}$ does not contain a point of K. Then, since

K is compact, there exists a finite number of open sets $\{V_i : i = 1,2,\cdots,n\}$ such that $K \subseteq \bigcup_{i=1}^{n} V_i$ and $V_i \cap F_i = \Phi$ for some $F_i \in \mathcal{F}$. We note that $V = \bigcup_{i=1}^{n} V_i$ and $\bigcap_{i=1}^{n} F_i$ are disjoint. Therefore V cannot contain a member of $\mathcal{F}$. This contradiction proves the lemma.

From this section we can see that compact sets in a topological space 'behave like points'.

3.4. Paracompact Spaces.

A. Definition. A collection $\mathcal{U}$ of subsets of X is said to be locally finite if and only if each $x \in X$ has a neighbourhood meeting only finitely many $u \in \mathcal{U}$.

LEMMA. (i) If $\{A_\lambda : \lambda \in I\}$ is a locally finite system of sets in X, then $\{\overline{A}_\lambda : \lambda \in I\}$ is locally finite.

(ii) If $\{A_\lambda : \lambda \in I\}$ is a locally finite system of sets in X, then $\bigcup \overline{A}_\lambda = \overline{\bigcup A_\lambda}$.

B. Definition. A T_2-Space X is said to be paracompact if and only if every open cover of X has an open locally finite refinement that 'covers' X.

It is clear that every compact T_2-Space is paracompact.

THEOREM. Let X be a T_3-Space. Then the following are equivalent:

(i) X is paracompact,

(ii) Every open cover of X has an open σ-locally finite (countable union of locally finite collections) refinement,

(iii) Every open cover of X has a locally finite refinement,

(iv) Every open cover of X has a closed locally finite refinement.

Proof. See [SW - (23)] - Page 146.

4. Stone-Čech Compactification.

Most of the hard examples in our work are based on subspaces of the Stone-Čech Compactification of a discrete space X. Therefore in this section we shall give some important properties of Stone-Čech Compactification of an arbitrary $T_{3\frac{1}{2}}$-Space X, which we denote by $\mathcal{B}X$.

4.1. Filter Description of $\mathcal{B}X$.

Let X be a $T_{3\frac{1}{2}}$-Space. Then we know that fixed Z-ultrafilters in X are in one to one correspondence with the points of X itself. We shall fix all the free Z-ultrafilters by adding points to X. This enlarged set is the Stone-Čech Compactification of X and we denote by $\mathcal{B}X$ (βX).

The points of $\mathcal{B}X$ are in one to one correspondence with the Z-ultrafilters on X and we write the points of $\mathcal{B}X$ as $(A^p)_{p \in \mathcal{B}X}$ with the understanding that for $p \in X$, A^p converges to p.

We write $\overline{Z} = \{p \in \mathcal{B}X : Z \in A^p\}$; $\mathcal{B}X$ is made into a topological space by taking the family of all sets $\overline{Z}$ as a base for the closed sets where Z is a zero set of X.

4.2. Some Properties of $\mathcal{B}X$.

A. LEMMA. Let Z be a zero set of X, then $\overline{Z}$ defined above is

$cl_{\beta X} Z$.

<u>Proof.</u> By the definition of the topology of βX, $cl_{\beta X} Z \subseteq \overline{Z}$.

Let $\overline{Z}_1$ be a basic closed set containing Z. Then $Z_1 = \overline{Z}_1 \cap X \supset Z$ and hence $\overline{Z}_1 \supset \overline{Z}$. Therefore $cl_{\beta X} Z \supset \overline{Z}$. This completes the proof of the lemma.

B. <u>Note.</u> (i) $\overline{X} = \beta X$,

(ii) $cl_{\beta X} X = \beta X$,

(iii) $p \in cl_{\beta X} Z$ if and only if $Z \in A^p$,

(iv) Let $p \in X$. Then $Z(M_p) = A_p$ where $M_p = \{f \in C(X) : f(p) = 0\}$. (We shall give the following well-known theorem [see - (3)] which is useful in the latter chapters.)

C. <u>THEOREM.</u> Every $T_{3\frac{1}{2}}$ space X has a compactification βX with the following equivalent properties:

(i) Every continuous mapping τ from X into any compact space y has a continuous extension $\overline{\tau}$ from βX into y,

(ii) Every function f in $\mathscr{C}^*(X)$ has a continuous extension to βX,

(iii) For any two zero sets Z_1 and Z_2 in X, $cl_{\beta X}(Z_1 \cap Z_2) = cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2$,

(iv) Distinct Z-ultrafilters on X have distinct limits in βX.

D. <u>Note.</u> If T is any other compactification of X which satisfies any one of (i)-(iv), then βX is homeomorphic T. Therefore

$\mathcal{B}X$ of a space X is essentially unique.

4.3. Some Applications.

A. THEOREM. Any product of compact T_2-spaces is compact.

Proof. Let $X = \prod_{i \in I} X_i$ and each X_i be a compact T_2-space. The projection map $\Pi_i : X \to X_i$ is continuous and therefore there exists a continuous extension $\overline{\Pi}_i : \mathcal{B}X \to X_i$. Let $h : \mathcal{B}X \to X$ where $h(x) = (\overline{\Pi}_i(x))$. Then h is continuous and onto. Hence X is compact.

Note. This is the Tychonoff's product theorem for T_2-spaces.

B. Definition. Let S be a subspace of the topological space X. Then S is said to be C-embedded in X if every function in $C(S)$ can be extended to a function in $C(X)$ and S is said to be $\mathcal{C}^*$- embedded in X if every function in $\mathcal{C}^*(S)$ can be extended to a function in $\mathcal{C}^*(X)$.

C. Note. If a function f in $\mathcal{C}^*(S)$ has an extension g in $C(X)$, then f also has a bounded extension.

D. LEMMA. Let X be a $T_{3\frac{1}{2}}$-space. Then we have the following:

(i) A subspace S of X is $\mathcal{C}^*$-embedded in X if and only if $cl_{\mathcal{B}X}\, S = \mathcal{B}S$,

(ii) Every compact set in X is $\mathcal{C}^*$-embedded in X,

(iii) If S is open and closed in X, the $cl_{\mathcal{B}X}\, S$ and $cl_{\mathcal{B}X}\,(X-S)$ are complementary open sets in $\mathcal{B}X$,

(iv) An isolated point of X is isolated in $\mathcal{B}X$ and X is open in $\mathcal{B}X$ if and only if X is locally compact.

4.4. Some Examples.

A. The Space $\beta\mathbb{N}$. ($\mathbb{N}$ = set of all positive integers)

(i) Every point of $\mathbb{N}$ is an isolated point. Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, $\mathbb{N}$ is the set of all isolated points of $\beta\mathbb{N}$,

(ii) The space $\mathbb{N}$ is locally compact and therefore $\mathbb{N}$ is an open subspace of $\beta\mathbb{N}$,

(iii) Let S be a subset of $\mathbb{N}$. Then $cl_{\beta X} S$ is both open and closed in $\beta\mathbb{N}$,

(iv) Let τ be a mapping from $\mathbb{N}$ onto Q. Then there exists a continuous extension $\bar{\tau} : \beta\mathbb{N} \to \beta Q$. Since Q is dense in βQ, $\bar{\tau}(\beta\mathbb{N}) = \beta Q$. It is trivial that $\mathbb{N}$ is $\mathscr{C}^*$-embedded in $\mathbb{R}$. Therefore $\mathbb{N}$ is $\mathscr{C}^*$-embedded in Q and hence $cl_{\beta Q}\, \mathbb{N} = \beta\mathbb{N}$.

B. The Space $\beta\mathbb{R}$. ($\mathbb{R}$ = set of all reals)

$\mathbb{R}$ is locally compact and hence $\mathbb{R}$ is open $\beta\mathbb{R}$. Let τ be a mapping from $\mathbb{N}$ onto $\mathbb{R}$. Then there exists a continuous extension $\bar{\tau} : \beta\mathbb{N} \to \beta\mathbb{R}$. Then it follows that, $\tau(\beta\mathbb{N}) = \beta\mathbb{R}$. It is easy to see that $cl_{\beta\mathbb{R}}\, \mathbb{N} = \beta\mathbb{N}$.

C. Cardinals of the Spaces $\beta\mathbb{N}$, βQ and $\beta\mathbb{R}$: (Q = set of all rationals)

In the sub-section 4.4-A we proved that $\bar{\tau}(\beta\mathbb{N}) = \beta Q$ and $cl_{\beta Q}\, \mathbb{N} = \beta\mathbb{N}$. Hence we have $|\beta\mathbb{N}| \geq |\beta Q|$ and $|\beta Q| \geq |\beta\mathbb{N}|$. Therefore $\beta\mathbb{N}$ and βQ have the same cardinality. From the sub-section 4.4-B, we can see that $\beta\mathbb{N}$ and $\beta\mathbb{R}$ have the same cardinality. Hence all three spaces $\beta\mathbb{N}$, βQ and $\beta\mathbb{R}$ have the same cardinality.

D. A Discrete Space X.

Let X be an infinite discrete space. Then we have the following:

(i) For $V \subset X$, $cl_{\mathcal{B}X} V$ is an open and closed subset of $\mathcal{B}X$,

(ii) For $S \subset X$, $cl_{\mathcal{B}X} S = \mathcal{B}S$,

(iii) $|\mathcal{B}X| = 2^{2^{|X|}}$,

(iv) $|cl_{\mathcal{B}X} S| = 2^{2^{|S|}}$ where $S \subset X$.

As a special case of the above results we have $|\mathcal{B}\mathbb{N}| = 2^C$ where $C = 2^{\aleph_0}$ and hence $|\mathcal{B}\mathbb{N}| = |\mathcal{B}Q| = |\mathcal{B}\mathbb{R}| = 2^C$. Moreover $|\mathcal{B}\mathbb{N} - \mathbb{N}| = |\mathcal{B}Q - Q| = |\mathcal{B}\mathbb{R} - \mathbb{R}| = 2^C$.

We shall begin our study of m-n compact spaces in the next chapter.

CHAPTER II

m-n COMPACT SPACES

1. Generalized Topological Notions

The property compactness and other variations of this concept are special cases of the more general form m-n compactness. In this section we give some basic definitions and study some basic properties of m-n compact spaces.

1.1. Compactness.

A. Definition. Let X be a topological space. An m-Fold open cover of X is a collection $\{u_i : i \in I\}$ of open subsets of X such that $X = \bigcup_{i \in I} u_i$ and $|I| = m$.

B. Definition. Let m and n be infinite cardinals and $m \geq n$. A space S is said to be m-n compact if and only if every m-Fold open cover of X has a subcover of cardinality strictly less than n.

C. Definition. A space X is said to be n-compact if it is m-n compact for each m.

Some authors use the terminology, finally n-compact for n-compact and some use the symbol ∞ for m when there is no restriction on the cardinality of the initial cover.

D. Special Cases.

(i) $\aleph_o$-compact spaces $\equiv$ compact spaces,

(ii) $\aleph_o$- $\aleph_o$ compact spaces $\equiv$ countably compact spaces,

(iii) $\aleph_1$- compact spaces $\equiv$ Lindelöf spaces,

(iv) $m - \aleph_0$ compact spaces $\equiv$ initially m-compact spaces.

1.2. Character of a Space.

A. Definition. A space is said to have character $\leq m$ if every point in the space has a neighbourhood base of cardinality less than or equal to m.

B. Examples.

(i) A first countable space is a space with character $\leq \aleph_0$.

(ii) A discrete space is a space with character 1.

note. The character of a space is a local property. In the next sub-section we shall study the stability of the neighbourhood system at a point.

1.3. Stable Local Bases.

A. Definition. A collection τ of non-empty sets is said to be $<$ m-stable if for each $\tau' \subset \tau$ with $|\tau'| < m$, there exists a $G \in \tau$ such that $\bigcap \tau' \supseteq G$.

B. Definition. A space X is said to be $<$ m-discrete if and only if every point of X has a $<$ m-stable neighbourhood base.

C. Examples.

(i) Every discrete space is $<$ m-discrete for any cardinal number m.

(ii) Let X be a $<$ m-discrete space and τ' be a collection of open subsets of X. If $|\tau'| < m$, then $\bigcap \tau'$ is open.

Note.

(i) Every $\mathcal{G}_\delta$-set in an $<$ m-discrete space is open where m is an uncountable cardinal.

(ii) Every topological space is $< \aleph_0$-discrete.

D. Definition. A space X is said to be a P-space if every prime ideal in $C(X)$ is maximal.

LEMMA. In a $T_{3\frac{1}{2}}$-space, every $\mathcal{G}_\delta$-set containing a compact set S contains a zero-set which contains S.

Proof. Straight forward.

THEOREM. Let X be a $T_{3\frac{1}{2}}$-space. Then the following are equivalent:

(i) X is a P-space,

(ii) Every zero-set in X is open,

(iii) Every $\mathcal{G}_\delta$-set in X is open,

(iv) X is $< \aleph_1$-discrete.

Proof. Use the Lemma and the definition.

1.4. Some Basic Properties of m-n Compact Spaces.

A. THEOREM. A continuous image of a m-n compact space is m-n compact.

B. COROLLARY. Let $X = \prod_{i \in I} X_i$ be a m-n compact space. Then every sub-product $X_{I'}$ of X is m-n compact. In particular every factor space of X is m-n compact.

C. THEOREM.

(i) A closed subspace of m-n compact space is m-n compact.

(ii) Let $\{X_i : i \in I\}$ be a collection of m-n compact subspaces of a space X. If $|I| < n$ and n is regular, then $\bigcup_{i \in I} X_i$ is m-n compact.

The above properties are similar to the properties of compact spaces. We shall give a trivial generalization of a well-known property of a compact set.

D. <u>THEOREM</u> Let X be a $< n$- discrete T_2- space and let S be a m-n compact subset of X with $|S| = m$. Then S is a closed subset of X.

2. Generalized Filters

Filters are convenient tools in the study of compactness and its other variations. Therefore in this section our main aim is to characterize m-n compact spaces using m-n Filters. Also some applications of Generalized Filters are included in this section.

2.1. m-n Filters.

A. <u>Definition.</u> A collection $\mathcal{F}$ of subsets of a set X has the <u>$<$ m-intersection</u> property if for each $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}| < m$, $\bigcap \mathcal{F}' \neq \Phi$.

<u>Note.</u> Every $< m$-stable collection of non-empty subsets of X has $< m$-intersection property.

B. <u>Definition.</u> A m-n Filter on a set X is a filter on X which has $< n$-intersection property and has a base $\mathcal{F}_B$ of cardinality less than or equal to m.

C. <u>Examples.</u>

(i) Let $X = \mathbb{R}$ (set of all reals) and $\mathcal{F} = \mathcal{V}_X$ (neighborhood system at $X \in X$). Then $\mathcal{F}$ is a m-n Filter on X where m and n are infinite cardinals.

We note that $\bigcap \mathcal{F} = \{x\}$ and therefore $\mathcal{F}$ is a fixed m-n filter.

It is easy to see that $\mathcal{F}$ is not $<$ n-stable for $n > \aleph_0$,

(ii) Let X be an infinite set with $|X| = m$ and S be an infinite subset of X with $|S| = n$. Let $P^{\aleph_0}(S)$ denote $\{A \in P(S) : |S-A| < \aleph_0\}$. Then $P^{\aleph_0}(S)$ is a filter base on X with $|P^{\aleph_0}(S)| = |S| = n$. Suppose n is regular, then $P^{\aleph_0}(S)$ has the $<$ n-intersection property. Let $\mathcal{F}$ be the filter generated by Then $\mathcal{F}$ is an m-n filter. We note that $\bigcap P^{\aleph_0}(S) = \Phi$ and therefore $\mathcal{F}$ is a free m-n filter.

D. Definition. An m-n Stable Filter on a set X is a filter on X which is $<$ n-stable and has a base $\mathcal{F}_B$ of cardinality less than or equal to m.

Example. Let X be a $<$ n -discrete space with character less than or equal to m. Let $\mathcal{V}_x$ be the neighbourhood system at x. Then $\mathcal{V}_x$ is a m-n stable filter.

LEMMA. Let $\mathcal{F}$ be an m-n filter on X. Suppose $m^{\overline{n}} = m$ and n is regular, then there exists a m-n stable filter $\mathcal{G}$ on X such that $\mathcal{G} \supset \mathcal{F}$.

Proof. Let $\mathcal{F}_B$ denote the filter base for $\mathcal{F}$ and let $\mathcal{G}_B = \{\bigcap \mathcal{F}' : \mathcal{F}' \subset \mathcal{F}_B, |\mathcal{F}'| < n\}$. Then $|\mathcal{G}_B| \leq \sum_{\alpha<n} m^{\alpha} = m^{\overline{n}} = m$ (By hypothesis) and since n is regular, $\mathcal{G}_B$ is $<$ n-stable. Let $\mathcal{G}$ be the filter generated by $\mathcal{G}_B$. Then $\mathcal{G}$ is an m-n stable filter and it is easy to see that $\mathcal{G} \supset \mathcal{F}$.

Notation. For filter bases $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \succ \mathcal{F}_2$ if the filter generated by $\mathcal{F}_1$ contains the filter generated by $\mathcal{F}_2$. We note that in the above Lemma $\mathcal{G}_B \succ \mathcal{F}_B$.

2.2. Filter Characterization of m-n Compactness

A. THEOREM. Let X be a topological space. Then the following are equivalent:

(i) X is m-n compact,

(ii) Every family of closed subsets of X with the $< n$-intersection property also has the $\leq m$-intersection property,

(iii) Every m-n filter on X has an adherent point.

Proof. (i) $\Rightarrow$ (ii) Let $\{F_i\}$ be a family of closed subsets of X with the $< n$-intersection property. Then $\{X - F_i\}$ does not contain a m-fold open cover of X and hence $\{F_i\}$ has $\leq m$-intersection property.

(ii) $\Rightarrow$ (iii) Let $\mathcal{F}$ be a m-n filter on X and let $\mathcal{F}_B$ be a filter base of $\mathcal{F}$. Then $\overline{\mathcal{F}}_B = \{\overline{A} : A \in \mathcal{F}_B\}$ has $< n$-intersection property and hence $\bigcap \overline{\mathcal{F}}_B \neq \Phi$. Therefore $\mathcal{F}$ has an adherent point in X.

(iii) $\Rightarrow$ (i) Suppose X is not m-n compact. Then there exists a m-fold open cover $\{G_i : i \in I\}$ of X with no sub-cover of cardinality less than n. Hence $\{X - G_i : i \in I\}$ is a filter sub-base for some filter base $\mathcal{F}_B$ (say). The filter $\mathcal{F}$ generated by $\mathcal{F}_B$ is a m-n filter and $\bigcap \overline{\mathcal{F}} = \bigcap_{i \in I} X - G_i = \Phi$. We have a contradiction and hence the space X is m-n compact.

B. COROLLARY. If n is regular and $m^{\overline{n}} = m$, then a topological space X is m-n compact if and only if every m-n stable filter on X has an adherent point.

2.3. α- Stable Filters

A. Definition. Let α be an infinite cardinal. A filter $\mathcal{F}$ on X which is α-stable is said to be an α-stable filter.

B. THEOREM. Let X be an infinite discrete space such that $|X|^{\underline{\alpha}} = |X| \geq \alpha \geq \aleph_0$. Then there is a family Φ_α of α-stable filters on X such that

(i) $|\Phi_\alpha| = 2^{2^{|X|}}$ and

(ii) If $\mathcal{F}_0, \mathcal{F}_1 \in \Phi_\alpha$ and $\mathcal{F}_0 \neq \mathcal{F}_1$, then there is a $A \in P(X)$ such that $A \in \mathcal{F}_0$ and $X - A \in \mathcal{F}_1$.

Proof. [W.W. Comfort - (3) - Page 146]

C. COROLLARY. Let X be an infinite discrete space, then $|\mathcal{B}(X)| = 2^{2^{|X|}}$.

Proof. We note that $\mathcal{B}X$ and the set of all maximal filters on X are in one to one correspondence. By the above Theorem, there is a $\Phi_{\aleph_0}$ such that,

(i) $|\Phi_{\aleph_0}| = 2^{2^{|X|}}$ and ,

(ii) $\mathcal{F}_0, \mathcal{F}_1 \in \Phi_{\aleph_0}$ are contained in different maximal filters if $\mathcal{F}_0 \neq \mathcal{F}_1$. By (ii), $|\mathcal{B}X| \geq |\Phi_{\aleph_0}|$ and hence $|\mathcal{B}X| \geq 2^{2^{|X|}}$.

Therefore $|\mathcal{B}X| = 2^{2^{|X|}}$.

D. Note: An α-stable filter $\equiv \infty - \alpha$ Stable Filter in the terminology of the sub-section 2.1 of this chapter.

2.4. Some Applications

A. LEMMA. Let X be a $<$ n-discrete, T_2-space and let A be an

n-compact subset of X. Then $A = \bigcap \overline{W}$ where W is an open neighbourhood of A.

Proof. Suppose $x \notin A$. Then we note that there exist two open subsets U and V of X such that $A \subset U$, $x \in V$ and $U \cap V = \Phi$. Hence $x \notin \overline{U}$. This proves the Lemma.

B. THEOREM. Let X be an m-n compact space and let $\mathcal{F}$ be an m-n stable filter base where $m^{\overline{n}}$ and n is regular. Then the following hold:

(i) For any open subset V of X containing $ad_X \mathcal{F}$, there exists a $F \in \mathcal{F}$ such that $V \supset F$,

(ii) If $ad_X \mathcal{F}$ is n-compact and X is a subspace of a $<$ n-discrete T_2-space Y, then $ad_X \mathcal{F} = ad_y \mathcal{F}$.

Proof. (i) Suppose V does not contain any $F \in \mathcal{F}$. Then $X - V \cap F \neq \Phi, \forall F \in \mathcal{F}$. Hence $\{X - \underline{V \cap F} : F \in \mathcal{F}\}$ is a m-n stable filter base on X. We note that $\bigcap X - V \cap F = \Phi$. Hence we have a contradiction.

(ii) By Lemma A, $ad_x \mathcal{F} = \overline{W}$ where W is an open neighbourhood of $ad_x \mathcal{F}$ in Y. By (i) each W contains a member of $\mathcal{F}$ and hence $ad_y \mathcal{F} \subseteq ad_x \mathcal{F}$. Therefore $ad_x \mathcal{F} = ad_y \mathcal{F}$.

C. Definition. A space X is said to be strongly m-n compact if for every m-n stable filter base $\mathcal{F}$ on X there exists a compact set K of X such that $\mathcal{F}/_K$ is a m-n stable filter base.

Note. For regular n and $m^{\overline{n}} = m$, the property strong m-n compactness is stronger than m-n compactness.

D. <u>Notation.</u> $\mathcal{C}^*$ = Family of spaces whose every infinite subset meets some compact subset in an infinite set.

<u>LEMMA.</u> (i) A space X belongs to $\mathcal{C}^*$ if and only if X is strongly $\aleph_0 - \aleph_0$ compact.

(ii) Every sequentially compact space belongs to $\mathcal{C}^*$.

<u>Proof.</u> (i) '$\Rightarrow$' Let $\mathcal{F}$ be a $\aleph_0 - \aleph_0$ filter base on X. Let $\mathcal{F}' = \{F_n' : F_n' = \bigcap_{\alpha=1}^{n} F_\alpha,\ F_\alpha \in \mathcal{F}\}$ and pick x_n from F_n' for every $n = 1,2,\cdots$. Then there exists a compact set S such that S contains infinitely many elements of $\{x_n\}$. Hence $F_n \cap S \neq \Phi$ for every $n = 1,2,\cdots$. Therefore $\mathcal{F}/S$ is a filter base and $|\mathcal{F}/S|$ is less then or equal to $\aleph_0$.

'$\Leftarrow$' Let A be an infinite subset of X. Take $\{x_n : n = 1,2,\cdots\}$ from A. Let $B_k = \{x_n : n \geq k\}$. Then $\mathcal{F} = \{B_k\}$ is a $\aleph_0 - \aleph_0$ filter base on X. By hypothesis there exists a compact set S such that $B_k \cap S \neq \Phi$ for all $k = 1,2,\cdots$. Hence $|A \cap S| \geq \aleph_0$.

(ii) Let X be a sequentially compact space and let A be an infinite subset of X. Consider $\{x_n\}_{n=1}^{\infty}$ in A. Then there exists a convergent subsequence $\{x_{n_k}\}_{n=1}^{\infty}$. Suppose $x_{n_k} \to x$ as $k \to \infty$. Let $S = \{x_{n_k}\}_{k=1}^{\infty} \cup \{x\}$. Then $A \cap S$ is infinite and S is a compact subset of X. Hence $X \in \mathcal{C}^*$.

<u>Note.</u> It is easy to see that every compact space is strongly $\aleph_0$-$\aleph_0$ compact and by the previous Lemma every sequentially compact space is strongly $\aleph_0$-$\aleph_0$ compact.

3. Productivity of m-n Compactness

We know that arbitrary products of compact spaces are compact but many properties similar to compactness are not preserved by even two products. In this section we study sufficient conditions for products to have the generalized property m-n compactness.

3.1. The Spaces Which Satisfy $1_{m,n}$.

A. Definition. A topological space X said to satisfy the property $1_{m,n}$, if for every m-n filter base $\mathcal{F}$ on X there exists a compact set K and an m-n stable filter base $\mathcal{G}$ such that $\mathcal{G} > \mathcal{F}$ and $\mathcal{G} > V_K$ where V_K is the open neighbourhood base of K.

B. LEMMA. (i) Let n be a regular cardinal and $m^{\underline{n}} = m$. Suppose X is m-n compact, $<$ n-discrete and has character $\leq m$, then X satisfies $1_{m,n}$.

(ii) The property $1_{m,n}$ is stronger than m-n compactness.

(iii) Let n be regular and $m^{\underline{n}} = m$. Then every strongly m-n compact space satisfies $1_{m,n}$.

(iv) Let n be regular and $m^{\underline{n}} = m$, then every locally compact, m-n compact space is strongly m-n compact.

Proof. (i) Let $\mathcal{F}$ be an m-n filter base on X. Then there exists an m-n stable filter base $\mathcal{F}'$ on X such that $\mathcal{F}' > \mathcal{F}$. Let $x \in \bigcap \overline{\mathcal{F}'}$ and let V_x be an open neighbourhood base at x with $|V_x| \leq m$. Then take $\mathcal{G} = \mathcal{F}' \vee V_x$.

(ii) This is easy.

(iii) This is trivial.

(iv) Let $\mathcal{F}$ be a m-n stable filter base and let $x \in \bigcap \overline{\mathcal{F}}$. Let V be a compact neighbourhood of x. Then $\mathcal{F}/V$ is a m-n stable filter base and this proves (iv).

C. Note. Let n be regular and $m^{\overline{n}} = m$. Then we have the following implications for the topological space X:

$$\begin{array}{ccc} X \text{ is compact} & \longrightarrow & X \text{ is strongly m-n compact} \\ & & \downarrow \\ X \text{ is m-n compact} & \longleftarrow & X \text{ has the property } 1_{m,n} \end{array}$$

D. LEMMA. $1_{\aleph_0,\aleph_0}$ is equivalent strongly to $\aleph_0 - \aleph_0$ compactness.

Proof. Suppose X has the property $1_{\aleph_0,\aleph_0}$. Let $\mathcal{F}$ be a filter base on X such that $|\mathcal{F}| \le \aleph_0$. Then there exists a filter base $\mathcal{G}$ on X and a compact subset K of X such that $\mathcal{G} > \mathcal{F}$, $\mathcal{G} > V_K$ and $|\mathcal{G}| \le \aleph_0$ where V_K is the open neighbourhood base of K. Let $\mathcal{G} = \{G_1, G_2, \cdots\}$ and $G_n \supseteq G_{n+1}$ for $n = 1,2,\cdots$. Let $x_n \in G_n$ for $n = 1,2,\cdots$ and $S = \{x_n\}_{n=1}^{\infty} \cup K$. Then S is compact and $\mathcal{F}/S$ is a filter base on X. This proves the Lemma.

3.2. Initially m-Compact Spaces.

A. THEOREM [JEV -(18)] Let $X = \prod_{i \in I} X_i$ where each X_i satisfies $1_{m,\aleph_0}$. Then we have the following:

(i) If $|I| \le m$, then X satisfies $1_{m,\aleph_0}$,

(ii) If $|I| \le m^+$, then X is initially m-compact.

This is one of the attractive and strong theorem in the section of m-n compactness and it has beautiful consequences which are well known and hard theorems.

B. <u>COROLLARY.</u> (i) Every product of at most $\aleph_1$ strongly $\aleph_o - \aleph_o$ compact spaces is countably compact.

(ii) Every product of at most $\aleph_1$ sequentially compact spaces is countably compact.

(iii) Every product of at most m^+ initially m-compact spaces of character less than or equal to m is initially m-compact.

(iv) Strong $\aleph_o - \aleph_o$ compactness is countably productive.

<u>Proof.</u> (i) Follows from the Theorem <u>A</u> - (ii)

(ii) Follows from (i).

(iii) Follows from the fact that if X is initially m-compact and has character less than or equal to m, then X has the property $1_{m,\aleph_o}$.

(iv) This follows from 3.1 - <u>D</u>.

3.3. <u>m-n Compact Spaces.</u>

A. <u>THEOREM.</u> Let X be a m-n compact space and suppose Y has the property $1_{m,n}$ where n is regular and $m^{\overline{n}} = m$. Then $X \times Y$ is m-n compact.

<u>Proof.</u> Let $\mathcal{F}$ be a m-n stable filter base on $X \times Y$ and let Π_2 be the projection map on to Y. Then $\Pi_2(\mathcal{F})$ is a m-n stable filter base on Y and since Y has the property $1_{m,n}$, there exist a m-n stable filter base $\mathcal{G}$ on Y and a compact subset K of Y such that

$\mathcal{G} > \Pi_2(\mathcal{F})$ and $\mathcal{G} > V_K$ where V_K is the open neighbourhood base of K. We note that $\mathcal{F}' = \mathcal{F} \vee \Pi_2^{-1}(\mathcal{G})$ is a m-n stable filter base on $X \times Y$ and since X is m-n compact, $\bigcap \Pi_1(\overline{\mathcal{F}})$ is non-empty where Π_1 is the projection map on to the space X. Let $x \in \bigcap \Pi_1(\overline{\mathcal{F}})$ and V be a neighbourhood of x. Then $(V \times G) \cap F \neq \Phi$ for every $G \in \mathcal{G}$ and $F \in \mathcal{F}$. We know that $\mathcal{G} > V_K$ and since K is compact, $(\{x\} \times K) \cap (\bigcap \overline{\mathcal{F}}) \neq \Phi$ and hence $\bigcap \overline{\mathcal{F}} \neq \Phi$.

B. <u>COROLLARY.</u> (i) Let X be a space with the property $1_{m,\aleph_0}$ and suppose that Y is an initially m-compact space. Then $X \times Y$ is initially m-compact.

(ii) If each X_i is initially m-compact for all $i \in I$ where $|I| \leq m$ and all but one have character $\leq m$, then $X = \prod_{i \in I} X_i$ is initially m-compact.

(iii) A product of not more than m initially m-compact and all but one locally compact spaces is initially m-compact.

(iv) Let n be regular and $m^{\overline{n}} = m$. Let X be a m-n compact space and Y be a strongly m-n compact space. Then $X \times Y$ is m-n compact.

<u>Proof.</u> (i) Take $n = \aleph_0$ in the Theorem <u>A</u> of this sub-section.

(ii) Follows from 3.2- A(i) and the (i) of this Corollary.

(iii) Follows from 3.1 - B(iv), B(iii) and 3.2 - A(i).

(iv) We note that for regular n and $m^{\overline{n}} = m$ strongly m-n compact spaces satisfy the property $1_{m,n}$.

C. <u>LEMMA.</u> Let $f_\alpha : X \to X_\alpha$. Let $\mathcal{F}$ be a m-n stable filter base on X and let $\mathcal{G}_\alpha$ be a m-n stable filter base on X_α. If

$\mathscr{F} \vee f_\alpha^{-1}(\mathscr{G}_\alpha)$ is filter sub-base on X, then $\mathscr{F} \vee f_\alpha^{-1}(\mathscr{G}_\alpha)$ is a m-n stable filter base on X.

Proof. Let $\mathscr{H} \subset \mathscr{F} \vee f_\alpha^{-1}(\mathscr{G}_\alpha)$ with $|\mathscr{H}| < n$. Let $H \in \mathscr{H}$, then $H = (\cap \mathscr{F}_H) \cap (f_\alpha^{-1}(\cap \mathscr{G}_H))$ where $\mathscr{F}_{\mathscr{H}} \subset \mathscr{F}$, $\mathscr{G}_H \subset \mathscr{G}_\alpha$ and $|\mathscr{F}_H| < \aleph_o$, $|\mathscr{G}_H| < \aleph_o$.

Let $\mathscr{F}' = \cup \{\mathscr{F}_H : H \in \mathscr{H}\}$ and

$\mathscr{G}' = \cup \{\mathscr{G}_H : H \in \mathscr{H}\}$.

We note that $|\mathscr{F}'| < n$ and $|\mathscr{G}'| < n$ and since $\mathscr{F}$ and $\mathscr{G}_\alpha$ are $< n$-stable, there exist $G \in \mathscr{G}_\alpha$ and $F \in \mathscr{F}$ such that $\cap \mathscr{G}' \supseteq G$ and $\cap \mathscr{F}' \supseteq F$. Therefore $\mathscr{F} \vee f_\alpha^{-1}(\mathscr{G}_\alpha)$ is $< n$-stable and it is easy to see that $|\mathscr{F} \vee f_\alpha^{-1}(\mathscr{G}_\alpha)| = |\mathscr{F}| \times |\mathscr{G}_\alpha| \leq m$. Hence the Lemma is proved.

We can now give a countable product Theorem using the condition $1_{m,n}$ which has a nice application to Lindelöf, $T_{3\frac{1}{2}}$, P-Spaces.

D. THEOREM. If X_i satisfies $1_{m,n}$ for $i = 1,2,\cdots$ and if n is regular and $m^{\underline{n}} = m$, then $X = \prod \{X_i : i = 1,2,\cdots\}$ is m-n compact.

Proof. Let $\mathscr{F}$ be a m-n stable filter base on X and let $\Pi_k : X \to X_k$ for $k = 1,2,\cdots$. Then $\Pi_k(\mathscr{F})$ is a m-n stable filter base on X_k and since X_k satisfies $1_{m,n}$, there exists a m-n stable filter base $\mathscr{G}_k$ and a compact set S_k such that $\mathscr{G}_k > \Pi_k(\mathscr{F})$ and $\mathscr{G}_k > \mathbf{V}_{S_k}$ where $\mathbf{V}_{S_k}$ is the open neighbourhood base of S_k. We note that $\mathscr{F} \vee \Pi_k^{-1}(\mathscr{G}_k)$ is a filter sub-base on X and by the Lemma (C) of this section $\mathscr{F} \vee \Pi_k^{-1}(\mathscr{G}_k)$ is a m-n stable filter base on X. We shall prove by induction that $\mathscr{G} = \mathscr{F} \vee (\vee \{\Pi_k^{-1}(\mathscr{G}_k) : k = 1,2,\cdots\})$ is $< n$-

stable. Let $\mathcal{H} = \mathcal{F} \vee (\vee \{\Pi_k^{-1}(\mathcal{G}_k): k = 1,2,\cdots \ell\})$ and suppose $\mathcal{H}$ is $< n$-stable. It is easy to see that $|\mathcal{H}| \leq m$ and by the Lemma (C) of this section $\mathcal{H} \vee \Pi_{k+1}^{-1}(\mathcal{G}_{k+1})$ is a m-n stable filter base on X. Therefore by induction $\mathcal{G}$ is $< n$-stable. Let $S = \Pi S_i$, $i = 1,2,\cdots$ and it is clear that $\mathcal{G} > \mathcal{F}$ and $\mathcal{G} > \mathbf{V}_S$ (the open neighbourhood base of S). Hence $\mathcal{G}$ has a cluster point in S and therefore $\bigcap \bar{\mathcal{F}} \neq \Phi$. Hence we have the theorem.

Note. In the above proof $|\mathcal{G}| \leq m$ and hence X satisfies the property $1_{m,n}$. Therefore for regular n and $m^{\bar{n}} = m$ the property $1_{m,n}$ is countably productive.

COROLLARY. If X_i is m-n compact, $< n$ discret and has character less than or equal to m for $i = 1,2, \cdots n$ and if n is regular and $m^{\bar{n}} = m$, then $X = \Pi X_i$, $i = 1,2,\cdots$ is m-n compact.

3.4. n - Compact Spaces.

A. THEOREM. Let n be regular and suppose X_i is n-compact, $< n$-discrete for each $i = 1,2,\cdots$, then ΠX_i, $i = 1,2,\cdots$ is n-compact.

Proof. Follows from the above Corollary. (Taking m to be '∞')

COROLLARY. A countable product of Lindelöf, $T_{3\frac{1}{2}}$, P-Spaces is Lindelöf.

Proof. Take $n = \aleph_1$ and note that $T_{3\frac{1}{2}}$, P-Spaces are $< \aleph_1$-discrete.

B. Example. Let m be a singular cardinal which is a countable

sum of smaller infinite cardinals m_i. Let X_i be discrete spaces of cardinality m_i for all $i \in I$ where $m = \sum_{i \in I} m_i$ and $|I| = \aleph_0$. Let $X = I \times \prod_{i \in I} X_i$ and give discrete topology for I. We shall prove that X is not m-compact which shows that the regularity cannot be deleted from the hypothesis of the Theorem A.

Let $\pi_\alpha : X \to I$ and $V_i = \pi_\alpha^{-1}\{i\}$. Then $\{V_i : i \in I\}$ is a discrete cover of X by sets which are open and closed in X. Let $V_{i,x} = \{i\} \times \{x\} \times \prod_{d \neq i} X_d$ where $x \in x_i$. Then $\{V_{i,x} : X \in X_i\}$ is a discrete clopen collection of subsets of X with cardinality m_i. By taking one element from each $V_{i,x}$ form the set H_i; this H_i is a closed discrete subset of X and has cardinality m_i. Let $H = \bigcup_{i \in I} H_i$. Then we note that $|H| = \sum_{i \in I} m_i = m$ and H is a closed discrete subset of X. (By construction of H_i's). Hence X is not a m-compact space.

C. LEMMA. For a singular cardinal m every < m-discrete space is < m^+-discrete.

Proof. Let X be a < m-discrete space and let $\mathcal{G} = \{O_i : i \in I\}$ be a family of open subsets of X with $|I| = m$. Let α be the cofinality of m. Then there exists a partition $\{I_i : i \in A\}$ of I such that $|I| > |I_i|$ for all $i \in A$ where $|A| = \alpha$. Hence we have $\bigcap \mathcal{G} = \bigcap_{i \in A} (\bigcap \{O_i : i \in I_i\})$ and since $|I_i| < m$, $\bigcap \{O_i : i \in I_i\}$ is an open subset of X. Since m is singular, $\alpha < m$ and therefore $\bigcap \mathcal{G}$ is open in X. This proves that X is < m^+-discrete.

THEOREM. If each X_i is a T_1-space which is m-compact and < m-discrete

and if m is a singular cardinal which is not a countable sum of smaller cardinals, then $X = \prod X_i$, $i = 1,2,\cdots$ is m-compact.

<u>Proof.</u> If $Y \subset X_i$ and $|Y| = m$. Then, since X_i is T_1 and $< m^+$-discrete, Y is a closed subset of X_i. It is easy to see that Y is discrete and hence X_i is not m-compact. This is not true and therefore each X_i has cardinality less then m. Hence each X_i is discrete and has a base of cardinality $|X_i|$. We note that $\{\pi_i^{-1}\{x\} : x \in x_i, i = 1,2,\cdots\}$ is a sub-base for $\prod X_i$, $i = 1,2,\cdots$ and it has cardinality $\sum |X_i| < m$. Therefore X has a base of cardinality $< m$ and hence X is m-compact.

This theorem shows that the example -<u>B</u> of 3.4 works only for particular kind of singular cardinals.

D. <u>THEOREM.</u> (i) If X_i is m-compact for $i = 1,2,\cdots$ and all but one are $< m$-discrete, then $X = \prod X_i$, $i = 1,2,\cdots$ is m-compact for regular cardinal number m.

(ii) If X_i is m-compact for $i = 1,2,\cdots$ and all but one are locally compact, then for regular m, $X = \prod X_i$, $i = 1,2, \cdots$ is m-compact.

<u>Proof.</u> (i) Follows from the fact that $1_{\infty,m}$ is countably productive and the theorem 3.3-<u>A</u> for regular m.

(ii) Follows from the fact that locally compact m-compact spaces satisfy the property $1_{\infty,n}$ and the same reason as in (i).

<u>Note.</u> In the condition $1_{m,n}$, if there is no restriction on the cardinality m, we use the symbol $1_{\infty,n}$.

4. More on Productivity.

In this section we study the concept of m-Boundedness which is stronger than strong m-n compactness and we use this property to study m-n compactness on product spaces.

4.1. m-Bounded Spaces.

A. Definition. A space X is said to be m-Bounded if for every subset A of X with $|A| \leq m$, there exists a compact subset K of X such that $A \subseteq K$ where m is an infinite cardinal.

B. Examples. (i) Every compact space is m-Bounded for any m.

(ii) $\mathbb{R}$, Q, $\mathbb{N}$ are not even $\aleph_0$-Bounded.

C. Some Properties.

(i) The property m-Boundedness is closed hereditary.

(ii) If a space X is m-Bounded, then X is n-Bounded for every $n \leq m$.

(iii) Arbitrary product of m-Bounded spaces is m-Bounded.

(iv) For regular cardinal n and $m^{\overline{n}} = m$ we have the following relations:

$$\begin{array}{ccc} \text{m-Bounded} & \longrightarrow & \text{strongly } m\text{-}n \text{ compact} \\ & & \downarrow \\ & & m\text{-}n \text{ compact} \end{array}$$

In particular every m-Bounded space is initially m-compact.

D. THEOREM. Let Y be a compact T_1-space and let $X \subseteq Y$. If each point of Y-X has a $\leq m$-stable neighbourhood base in Y, then X is m-Bounded.

Proof. Let A be a subset of X with $|A| \leq m$. Let $y \in Y - X$ and since Y is T_1, for each $a \in A$, there exists an open set U_a in Y such that $y \in U_a$ and $a \notin U_a$. Since Y has a $\leq m$-stable neighbourhood base, there is an open set U_y in Y such that $y \in U_y$ and $U_y \cap A = \Phi$. Let $U = \bigcup \{U_y : y \in Y-X\}$ The U is open in Y and $A \subseteq Y-U = X-U$. Hence $X-U$ is compact and it contains A, as required.

We shall show that the concepts m-Bounded and m-n compact are closely related.

4.2. m-n Bounded Spaces.

A. Definition. Let m and n denote infinite cardinals. A space X is said to be m-n Bounded if for every subset A of X with $|A| \leq m$, there exists a n-compact subset G of X such taht $A \subseteq G$.

Note. (i) The concept of m-n Bounded is weaker than m-Bounded but stronger than m-n compact for regular n and $m^{\overline{n}} = m$.

(ii) Every n-compact space is m-n Bounded for any m and the property m-n Boundedness is closed hereditary.

B. THEOREM. Let X_i be $< n$ discrete and m-n Bounded for $i = 1,2,\cdots$. Then $X = \prod X_i$ is m-n Bounded.

Proof. Let A be a subset of X with cardinality $|A| \leq m$. Then $\Pi_i(A)$ is a subset of X_i with $|\Pi_i(A)| \leq m$ and since X_i is m-n Bounded, there exists a n-compact subset G_i of X_i such that $\Pi_i(A) \subseteq G_i$. Then $A \subseteq \prod G_i$ and we note that $G = \prod G_i$ is n-compact. This proves the Theorem.

C. LEMMA. Let X be a regular space with $|X| \leq m$. Then X is m-n compact if and only if every open m-n filter base has a cluster point.

Proof. '$\Rightarrow$' This follows from the fact that every open m-n filter base is a m-n filter base.

'$\Leftarrow$' Suppose X is not m-n compact. Then there exists a m-n filter base $\mathcal{F}$ such that $\bigcap \overline{\mathcal{F}} = \Phi$. Hence we have $X = \bigcup \{X - F : F \in \mathcal{F}\}$ and we note that for every $x \in X$ there exists an open set V_x such that $x \in V_x \subset \overline{V}_x \subset X - \overline{F}$ for some $F \in \mathcal{F}$. Let $\mathcal{G} = \{X - \overline{V}_x : x \in X\}$. Then $\mathcal{G}$ is an open m-n filter base on X with empty adherence. This contradicts the hypothesis. Hence X is m-n compact.

COROLLARY. Let X be regular and $< n$-discrete. Then X is n-compact if and only if every open filter base with $< n$-stable property has a non-empty adherence.

D. THEOREM. Let X be a regular, $< n$-discrete space and suppose X is m-n compact for regular n and $m^{\underline{n}} = m$ then X is ω-n Bounded for all ω for which $2^{\omega} \leq m$.

Proof. Let A be a subset of X with $|A| \leq \omega$. We shall prove that $\overline{A}$ is n-compact. Let $\mathcal{F}$ be an open filter base on $\overline{A}$ with $< n$-stable property. We note that $\mathcal{F}/A$ is a filter base on A with $< n$-stable property and $|\mathcal{F}/_A| \leq 2^{\omega} \leq m$. Therefore $\mathcal{F}/_A$ is a m-n stable filter base on X and since X is m-n compact, we have $ad_X(\mathcal{F}/_A) \neq \Phi$. Hence $\mathcal{F}$ has an adherent point in $\overline{A}$. This shows that

$\overline{A}$ is n-compact and therefore we have the Theorem.

COROLLARY. Let X be a regular, initially m-compact space. Let ω be a cardinal number such that $2^{\omega} \leq m$. Then X is ω-Bounded.

4.3. Some Applications.

A. LEMMA. Let m be a singular cardinal and suppose X is ω-n compact for all $\omega < n$. Then X is m-n compact.

Proof. Let $\mathcal{F}$ be a m-n filter base on X. We shall prove that $\bigcap \overline{\mathcal{F}} \neq \Phi$. We assume that $|\mathcal{F}| = m$. Let $\{\mathcal{F}_i : i \in I\}$ be a partition of $\mathcal{F}$ where $|I|$ = cofinality of m and $|\mathcal{F}_i| < m$ for all $i \in I$. Let $G_i = \bigcap \overline{\mathcal{F}}_i$ for all $i \in I$ and we note that $\{G_i : i \in I\}$ is a filter base on X with $<$ n-intersection property. It is easy to see that $\bigcap \overline{\mathcal{F}} = \bigcap \overline{G}_i$ and $|I| < m$. From this we get $\bigcap \overline{G}_i \neq \Phi$. Hence we have the result.

We use this Lemma to obtain two product Theorems for m-n compactness.

B. THEOREM. Let m and n be infinite cardinals such that m is singular and n is regular. Suppose $2^{\omega} \leq m$ and $\omega^{\underline{n}} = \omega$ for all ω such that $m > \omega \geq n$. Let X_i be a regular, $<$ n-discrete, m-n compact space for $i = 1,2,\cdots$, then $X = \prod X_i$, $i = 1,2,\cdots$ is m-n compact.

Proof. Let $m > \omega \geq n$ and then by Theorem D of the previous subsection, each X_i is ω-n bounded. Hence $\prod X_i$ is ω-n Bounded and therefore $\prod X_i$ is ω-n compact for all $\omega < m$. By the previous Lemma the product $X = \prod X_i$ is m-n compact.

C. THEOREM. Let m be a singular cardinal such that $2^{\omega} \le m$ for all $\omega < m$. Let X_i be an initially m-compact, regular space for all $i \in I$. Then $X = \prod_{i \in I} X_i$ is initially m-compact.

Proof. From the Corollary D of the previous section we note that each X_i is ω-Bounded and hence $X = \prod X_i$ is ω-Bounded. Therefore X is initially ω-compact for all $\omega < m$. Then by the Lemma A of this section we have the Theorem.

D. Note. From the above Theorem we can settle down the question of productivity for initially m-compact spaces for certain type of Cardinals m.

4.4. An Example. ['Folik' - Thro' (10)]

We wish to construct a space which is strongly $\aleph_0 - \aleph_0$ compact but not $\aleph_0$-Bounded. For this we need to note some elementary properties which we shall state below:

(i) Let ϕ be a one to one mapping from N onto Q. For each irrational number α select an increasing sequence (S_n) of rationals converging to d. Let $E_d = \{\phi^{-1}(S_n)\}$ and $\mathcal{E} = \{E_d\}$. Then $|\mathcal{E}| = 2^{\aleph_0}$ and $|E_d \cap E_{d'}| < \aleph_0$ for all $d \neq d'$.

(ii) Every infinite T_2 space contains a discrete subset with cardinality $\aleph_0$.

A. LEMMA. Let E be a subset of $\beta\mathbb{N}$ such that for any infinite subset D of $\beta\mathbb{N} - E$ there is an infinite subset I of D with $|E \cap c\ell_{\beta\mathbb{N}} I| < 2^{\aleph_0}$. Then $\beta\mathbb{N} - E$ belongs to the Class $\mathcal{C}^*$ (2.4-D).

Proof. Let I be a discrete subset of D with cardinality $\aleph_0$ such that $|E \cap cl_{\beta\mathbb{N}} I| < 2^{\aleph_0}$.

Let $\mathcal{G} = \{I_d \subset I : |I_d| = \aleph_0$ and $|I_d \cap I_{d'}| < \aleph_0$ for $d \neq d'\}$ and $|\mathcal{G}| = 2^{\aleph_0}$ (Use the property (i).) We note that $cl_{\beta\mathbb{N}} I = \beta I$ ($\because I \leftrightarrow N$) and let $I_d' = cl_{\beta I} I_d - I_d$ for $I_d \in \mathcal{G}$. Then since $|I_d \cap I_{d'}| < \aleph_0$, $I_d' \cap I_{d'}' = \Phi$ for all $d \neq d'$. Hence $I' \cap E = \Phi$ for some d. For suppose $E \cap I_d' \neq \Phi$ for all d, then E contains $2^{\aleph_0}$ elements of $cl_{\beta\mathbb{N}} I$ which contradicts the hypothesis.

Let $d = d_o$, for which $E \cap I_{d_o}' = \Phi$, then $I_{d_o}' \subset \beta\mathbb{N} - E$ and we know that $I_d \subset \beta\mathbb{N} - E$ for all d and hence $cl_{\beta I} I_{d_o} \subset \beta\mathbb{N} - E$. From this it follows that any infinite subset of $\beta\mathbb{N} - E$ contains an infinite subset (I_{d_o}) of a compact subset $(cl_{\beta I} I_{d_o})$. Hence $\beta\mathbb{N} - E$ belongs to the class $\mathcal{C}^*$.

B. Construction.

For each $x \in \beta\mathbb{N} - \mathbb{N}$, let $A_x = \beta\mathbb{N} - \{x\}$. Then taking $E = \{x\}$, $A_x \in \mathcal{C}^*$ for all $x \in \beta\mathbb{N} - \mathbb{N}$. (By A). Hence A_x is stongly $\aleph_0 - \aleph_0$ compact but we shall prove that A_x is not $\aleph_0$-Bounded for some $x \in \beta\mathbb{N} - \mathbb{N}$. Let $S = \Pi\{A_x : x \in \beta\mathbb{N} - \mathbb{N}\}$ and let $S_D = \{Y_n \in S : \Pi_x(Y_n) = n \in \mathbb{N}$ for every $x \in \beta\mathbb{N} - \mathbb{N}\}$ where $\Pi_x : S \longrightarrow A_x$. We note that S_D is a closed countably infinite, discrete subset of S and hence S is not $\aleph_0 - \aleph_0$ compact. Therefore S is not $\aleph_0$-Bounded which proves that A_{x_o} $(= \Pi_{x_o}(S))$ is not $\aleph_0$-Bounded for some $x_o \in \beta\mathbb{N} - \mathbb{N}$. This A_{x_o} is the candidate for our example.

Note. From the previous example we conclude that $\aleph_0$-Bounded

and strongly $\aleph_0 - \aleph_0$ compact spaces cover distinct classes.

5. Projection Maps and Initially m-Compact Spaces.

In this section we study necessary and sufficient conditions for projections to be closed and z-closed, using $m\text{-}\aleph_0$ compactness. In the last subsection we will give some applications to $m\text{-}\aleph_0$ compact spaces and products of sequentially compact spaces.

5.1. Closed Projections.

A. Definition. Let $\Pi_X : X \times Y \to X$. Then the projection map Π_X is said to be closed if Π_X maps closed sets to closed sets.

THEOREM. (i) Let $\Pi_X : X \times Y \to X$. If X is discrete, then Π_X is closed.

(ii) Let $\Pi_X : X \times Y \to X$ and let Y be a discrete space with cardinality n. Then Π_X is closed if and only if X is $\leq n$-discrete.

Proof. (i) is easy and for (ii) we note that

$\Pi_X(H) = \bigcup_{y \in y} \Pi_X((X \times \{y\}) \cap H)$ where $H \subset X \times Y$ and $\Pi_X : X \times \{y\} \to X$ is a homeomorphism.

B. LEMMA. Let $X = \prod_{i \in I} X_i$ with $|I| = m$ and let $H \subseteq X$ be $m - \aleph_0$ compact. If $U \supseteq H$ is open, then there exists a finite set $F \subseteq I$ and an open set $V \supseteq \Pi_F(H)$.

C. THEOREM. Let $X = \prod_{i \in I} X_i$ where $|I| = m$ and let $H = \prod_{i \in I} H_i$ where each $H_i \subseteq X_i$. Each projection $\prod_{i \in B} H_i \times X_{I-B} \to X_{I-B}$ is closed if and only if H is $m - \aleph_0$ compact for each finite $F \subset I$ and for

each $B \subset F$, the projection $\prod_{i\in B} H_i \times X_{F-B} \to X_{F-B}$ is closed.

Proof. [(9)-Page 171]

D. COROLLARY. Let $X = \prod_{i\in I} X_i$ where $|I| = m$. Each projection $X \to X_{I-\{i\}}$ is closed if and only if X is m-$\aleph_0$ compact and for each finite set $F \subset I$, each projection $X_F \to X_{F-\{i\}}$ is closed.

Proof. In Theorem (C), take $H_i = X_i$ for all $i \in I$ and $B = \{i\}$.

5.2. Z-Closed Projections.

A. Definition. Let $\Pi_X : X \times Y \to X$. Then the projection map Π_X is said to be Z-closed if Π_X maps zero-sets to closed sets.

THEOREM. Let $\Pi_X : X \times Y \to X$ and let Y be a discrete space with cardinality n. If X is completely regular, then Π_X is Z-closed if and only if X is $\leq$ n-discrete.

Proof. '$\Rightarrow$' We note that n-Fold union of zero-sets of X is closed and zero-sets form a base for the closed sets in X.

'$\Leftarrow$' Easy. (zero-sets are closed sets)

B. Definition. A space X is said to be pseuodocompact if and only if $C(X) = C^*(X)$

LEMMA. Every countably compact space is pseuodocompact.

C. THEOREM. Let $X = \prod_{\alpha\in A} X_\alpha$ be completely regular with infinite A. Then every $\Pi_\alpha : X \to X_\alpha$ for $\alpha \in A$ is z-closed if and only if X is pseuodocompact.

Proof. [(9) - Page 169]

D. <u>LEMMA.</u> A projection on a $T_{3\frac{1}{2}}$ Space is closed if and only if it is z-closed.

<u>Proof.</u> '$\Rightarrow$' Obvious

'$\Leftarrow$' Let $X \times Y$ be a T_4-space and let $\Pi_X : X \times Y \to X$. Let A be a closed subset of $X \times Y$. We shall prove that $\Pi_X(A)$ is closed. Suppose $x \notin \Pi_X(A)$, then for each $y \in Y$, there exist zero-set Z_y in $X \times Y$ such that $Z_y \supset A$ and $X \notin Z_y$. Hence $\Pi_X(A) \subseteq \bigcap_{y \in Y} \Pi_X(Z_y)$ and since Π_X is z-closed, $G = \bigcap_{y \in y} \Pi_X(Z_y)$ is closed in X. Therefore X-G is a neighbourhood of x and disjoint from $\Pi_X(A)$; which proves that $\Pi_X(A)$ is a closed subset of X.

5.3. <u>T_4-Products.</u>

A. <u>LEMMA.</u> Let $X = \prod_{\alpha \in A} X_\alpha$ where $|A| = m$ and suppose X is m-$\aleph_o$ compact and each finite subproduct of X is T_4. Then X is T_4 provided each projection on X is closed.

<u>Proof.</u> Let H and H' be disjoint subsets of X. Then since H is m-$\aleph_o$ compact by Lemma 5.1-B, there exist a finite set $F \subset A$ and an open set $V \supseteq \Pi_F(H)$ such that $V \times X_{A-F} \subseteq X - H'$. Hence $\Pi_F(H)$ and $\Pi_F(H')$ are disjoint closed subsets of the T_4 space X_F. Therefore there exist disjoint open sets V_1, V_2 in X_F such that $H \subseteq \Pi_F^{-1}(V_1)$ and $H' \subseteq \Pi_F^{-1}(V_2)$. Hence $X = \prod_{\alpha \in A} X_\alpha$ is T_4.

B. <u>THEOREM.</u> Let $X = \prod_{\alpha \in A} X_\alpha$ where $|A| = m$ and suppose each finite sub-product of X is T_4. If X is m-$\aleph_o$ compact, then X is T_4.

<u>Proof.</u> Let F be a finite subset of A. Then X_F is T_4 and

all of its projections are z-closed and hence each projection on X_F is closed. By the Corollary 5.1-D each projection on X is closed. Hence by the Lemma A we have the Theorem.

C. THEOREM. Let $T = \mathbb{N}^A$ (product of uncountably many copies of $\mathbb{N}$). Then T is not normal.

[This is due to A.H. Stone - (14).]

COROLLARY. Let $X = \prod_{\alpha \in A} X_\alpha$ be a T_4-space. Then there exists a $S \subseteq A$ such that $|A-S| \leq \aleph_0$ and $\prod_{\alpha \in S} X_\alpha$ is countably compact.

Proof. If the conclusion is not true then X contains a closed subset homeomorphic to T. This contradicts the fact that T is not normal.

D. THEOREM. Let $X = \prod_{\alpha \in A} X_\alpha$ where $|A| = m$ and suppose X is T_4. Then there exists a S with $|A-S| \leq \aleph_0$ such that $\prod_{\alpha \in S} X_\alpha$ is m-$\aleph_0$ compact.

Proof. By Corollary C, there exists a S with $|A-S| \leq \aleph_0$ such that $\prod_{\alpha \in S} X_\alpha$ is $\aleph_0$ - $\aleph_0$ compact and hence $\prod_{\alpha \in S} X_\alpha$ is pseudo-compact. By Theorem 5.2-C each projection on $\prod_{\alpha \in S} X_\alpha$ is z-closed and hence by 5.2-D each projection on $\prod_{\alpha \in S} X_\alpha$ is closed and therefore by the Corollary 5.1-D, $\prod_{\alpha \in S} X_\alpha$ is m-$\aleph_0$ compact.

5.4. More On T_4-Products.

A. THEOREM. Let $X = \prod_{\alpha \in A} X_\alpha$ be a T_4-space and suppose each X_α satisfies the condition $1_{m,\aleph_0}$. Then X is initially m-compact.

Proof. If $|A| \le \aleph_o$, then we have the Theorem by Section 3.2 (A). Therefore we assume $|A| > \aleph_o$. Then by Theorem 5.3 - D there exists a S with $|A-S| \le \aleph_o$ and $\prod_{\alpha\in S} X_\alpha$ is initially m-compact. Let $X_{A-S} = \prod_{\alpha\in A-S} X_\alpha$, then since $|A-S| \le \aleph_o$, X_{A-S} satisfies $1_{m,\aleph_o}$. Hence $X = X_S \times X_{A-S}$ is initially m-compact.

Note. If the product is T_4 and each factor space has character $\le m$, then m-$\aleph_o$ compactness is arbitrarily productive.

B. THEOREM. If the product is T_4, then arbitrary product of sequentially compact spaces is countably compact.

Proof. We note that sequentially compact spaces satisfy $1_{\aleph_o,\aleph_o}$ and hence taking $m = \aleph_o$ in Theorem A we obtain this theorem.

6. γ-Weak Topological Sums of n-Compact Spaces.

In this section we shall prove that under suitable conditions on a family of spaces $\{X_i : i \in I\}$, their γ-weak topological sums are n-compact. As a particular case of this we have a simple application to the products of Lindelöf Spaces.

6.1. Some Terminology.

A. Some Notations.

(i) If I is a set we define the collection,

$P_\gamma(I) = \{J \in P(I) : |J| < \gamma\}$.

(ii) Throughout this section our γ-weak (topological) sum $\gamma\left(\prod_{i\in I} X_i\right)$ of the collection of spaces $\{X_i : i \in I\}$ is with respect to a fixed point $P = (P_i)$ in $X = \prod_{i\in I} X_i$.

B. Definition. Let $\mathscr{B}$ be a basis for the topological space X. Then X is said to be m-n compact with respect to $\mathscr{B}$ if for every cover $\mathscr{U} \in P(\mathscr{B})$ of X with $|\mathscr{U}| = m$, there exists a $\mathscr{V} \in P_n(\mathscr{U})$ such that $X = \bigcup \mathscr{V}$

C. Note. (i) Let $\mathscr{B}$ be a base for the topological space X and suppose X is m-n compact in the usual sense. Then X is m-n compact with respect to $\mathscr{B}$.

(ii) Let $\mathscr{B}$ be a base for the topological space X. Then the concepts of n-compactness and n-compactness with respect to $\mathscr{B}$ are equivalent.

D. Example. Let X be a discrete space with cardinality m. Let $\mathscr{B} = \{X\} \cup \{\{\alpha \in X\} : \alpha < m\}$ (Let $W(m)$ = least ordinal with cardinality m and then $W(m)$ is isomorphic to X.) This X is not m-n compact but "m-n compact with respect to $\mathscr{B}$". This is a partial justification for the reason to introduce the concept of m-n compactness with respect to a given base $\mathscr{B}$.

6.2. m-n Compact Spaces.

A. LEMMA. Let $m \geq n \geq cf(n) > |I|^{\overline{\gamma}} \geq \gamma \geq \aleph_0$ and $\alpha \geq \aleph_0$. Let $\left(\prod_{i \in I'} X_i\right)_\alpha$ be m-n compact for all $I' \subset I$ and $|I'| < \gamma$, then $\left(\gamma\left(\prod_{i \in I} X_i\right)\right)_\alpha$ is m-n compact.

Proof. Let $X(I') = \{X \in \prod_{i \in I} X_i : X_i = P_i$ for all $i \in I-I'\}$ Then we note that $\gamma\left(\prod_{i \in I} X_i\right) = \bigcup \{X(I') : I' \subset I, |I'| < \gamma\}$ and $(X(I'))_\alpha$ is homeomorphic to $\left(\prod_{i \in I'} X_i\right)_\alpha$. Since $|I|^{\overline{\gamma}} < cf(n)$, we have the Lemma using the fact that $\sum_{i \in \mathscr{B}} n_i < n$ where $n_i < n$ for all $i \in \mathscr{B}$

and $|\mathcal{B}| < cf(n)$.

B. Note. The sets $U = \prod_{i \in I} U_i$ where U_i is open in X_i and $|R(U)| < \alpha$ are elements of the canonical basis for $\left(\prod_{i \in I} X_i\right)_\alpha$. Let $Y \subset \prod_{i \in I} X_i$, then the canonical basis for $(Y)_\alpha$ consists of all sets of the form $U \cap Y$ (with U as above).

C. LEMMA. Let $m \geq n \geq cf(n) > |I|^{\overline{\gamma}} \geq \gamma \geq \aleph_o$ and $\alpha \geq \aleph_o$. Let $\left(\prod_{i \in I} X_i\right)_\alpha$ be m-n compact with respect to its canonical basis for all $I' \subset I$ and $|I'| < \gamma$, then $\left(\gamma\left(\prod_{i \in I} X_i\right)\right)_\alpha$ is m-n compact with respect to its cannonical basis. (This is a weaker version of A.)

D. THEOREM. Let $m \geq n \geq \gamma \geq \aleph_o$ and $n \geq \alpha \geq \aleph_o$ with n regular and strongly γ-inaccessible. Let $\{X_i : i \in I\}$ be a family of spaces such that $\left(\prod_{i \in I'} X_i\right)_\alpha$ is m-n compact for all $I' \in P_\gamma(I)$, then $\left(\gamma\left(\prod_{i \in I} X_i\right)\right)_\alpha$ is m-n compact with respect to the canonical basis.

Proof. [W.W. Comfort - (4) - Page 34]

Note. The above achievement of (4) in full generality is important to the study of m-n compactness and it has several 'by-products' which we will study in the next subsections.

6.3. n-Compact Spaces.

A. THEOREM. Let $n \geq \gamma \geq \aleph_o$ and $n \geq \alpha \geq \aleph_o$ with n regular and strongly γ-inaccessible. Let $\{X_i : i \in I\}$ be a family of spaces such that $\left(\prod_{i \in I'} X_i\right)_\alpha$ is n-compact for all $I' \in P_\gamma(I)$, then $\left(\gamma\left(\prod_{i \in I} X_i\right)\right)_\alpha$ is n-compact.

<u>Proof.</u> This follows from 6.2-<u>D</u> and 6.1-<u>C</u>.

B. <u>Example.</u> Let $n > \alpha = \gamma \geq \aleph_0$ and let $\mathcal{B}$ be an infinite cardinal such that $\alpha^{\mathcal{B}} \geq n$ and $\mathcal{B} < \alpha$. Let $\{X_i : i \in I\}$ be a family of discrete spaces such that $|X_i| = \alpha$ for all $i \in I$ and $|I| = \mathcal{B}$. We shall note the following:

(i) $\left(\gamma\left(\prod_{i\in I} X_i\right)\right)_\alpha = \left(\prod_{i\in I} X_i\right)_\alpha$,

(ii) $\left(\prod_{i\in I} X_i\right)_\alpha$ has a discrete topology ,

(iii) $\left|\prod_{i\in I} X_i\right| = \alpha^{\mathcal{B}} \geq n$,

(iv) $\left(\prod_{i\in I'} X_i\right)_\alpha$ has a base of cardinality $\leq \alpha \cdot |I'|$ $= \alpha$ for all $I' \in P_\gamma(I)$.

This example shows that strongly γ-inaccessible property of n cannot be deleted from the hypothesis of <u>A</u> and hence from the original version <u>D</u> of the previous section.

C. <u>COROLLARY.</u> Let $n > \gamma \geq \aleph_0$ and $n \geq \alpha \geq \aleph_0$ with n regular and strongly γ-inaccessible and let $\{X_i : i \in I\}$ be a family of spaces such that $|X_i| < n$ for all $i \in I$. Then $\left(\gamma\left(\prod_{i\in I} X_i\right)\right)_\alpha$ is n-compact.

<u>Proof.</u> We note that $\left|\prod_{i\in S} X_i\right| < n$ for all $S \in P_\gamma(I)$. (Using an equivalent form of the definition of strong γ-inaccessibility.) Hence by <u>A</u> we have the Corollary.

D. <u>Note.</u> (i) Every infinite cardinal is strongly $\aleph_0$-inaccessible.

(ii) The direct sum (weak sum) of the family of spaces $\{X_i : i \in I\}$ is $\aleph_0\left(\prod_{i\in I} X_i\right)$.

COROLLARY. Let α be an uncountable regular cardinal and let $\{X_i : i \in I\}$ be a family of spaces such that $|X_i| < \alpha$ for each $i \in I$. Then the direct sum of $\{X_i : i \in I\}$ is α-compact in the α-Box topology.

Proof. This is an easy deduction of the Corollary C by taking $n = \alpha$ and $\gamma = \aleph_0$.

In the next sub-section we shall give an application to the products of n-compact spaces.

6.4. An Application.

A. THEOREM. Let $\{X_i : i \in I\}$ be a family of spaces such that each is n-compact and <n-discrete. Let n be a regular cardinal number, then $\aleph_0 \left(\prod_{i \in I} X_i\right)$ is n-compact.

Proof. We note that n is strongly $\aleph_0$-inaccessible and by Theorem A (3.4), $\left(\prod_{i \in S} X_i\right)_{\aleph_0}$ (usual product topology) is n-compact for all $S \in P_{\aleph_0}(I)$. Hence by A-(6.3), taking $\gamma = \aleph_0$ and $\alpha = \aleph_0$ we have the Theorem.

B. LEMMA. Let X be a paracompact T_2-Space with a dense n-compact subspace. Then X is n-compact.

Proof. Let S be a dense n-compact subspace of X. Let $\mathcal{U}$ be an open cover of X. Since X is paracompact, there exists an open locally finite refinement $\mathcal{U}'$ of $\mathcal{U}$ such that $X = \bigcup \mathcal{U}'$. Using the regularity of X and since S is dense in X, we can find a sub-cover $\mathcal{U}''$ of $\mathcal{U}$ such that $|\mathcal{U}''| < n$. Hence $\mathcal{U}$ has a subcover of cardinality less than n, which proves the Lemma.

C. <u>THEOREM.</u> Let $X = \prod_{i \in I} X_i$ be a paracompact T_2-Space and suppose each X_i is n-compact and < n-discrete for a regular cardinal number n. Then X is n-compact.

<u>Proof.</u> We know by Theorem <u>A</u>, $\aleph_o\left(\prod_{i \in I} X_i\right)$ is n-compact and it is trivial that $\aleph_o\left(\prod_{i \in I} X_i\right)$ is a dense subset of $\prod_{i \in I} X_i$. Hence by Lemma <u>B</u> we have the Theorem.

D. <u>COROLLARY.</u> If the product is paracompact and T_2, then an arbitrary product of Lindelöf P-Spaces is Lindelöf.

<u>Proof.</u> We note that for $T_{3\frac{1}{2}}$-Spaces the property of being a P-Space and < $\aleph_1$-discrete are quivalent. Hence by Theorem <u>C</u>, taking $n = \aleph_1$, we have the Corollary which extends the result that a countable product of Lindelöf P-Spaces is Lindelöf (3.4-<u>A</u>), partially.

7. <u>Construction of Examples.</u>

The main aim of this section is to prove the existence of a space which is m-$\aleph_o$ compact but does not satisfy $l_{m,\aleph_o}$. Hence we deduce that the property $l_{m,n}$ is strictly stronger than m-n compactness.

7.1. <u>A Space Which Does Not Satisfy</u> $l_{m,\aleph_o}$.

A. <u>Definition.</u> Let m be an infinite cardinal then a filter base $\mathcal{F}$ on a set X is said to be of type m if $|\mathcal{F}| \leq m$ and $|F| = m$ for all F in $\mathcal{F}$.

<u>Example.</u> Let $X = \mathbb{N}$, $\mathcal{F} = \{\mathbb{N} - A : |A| < \aleph_o\}$. Then $\mathcal{F}$ is a Filter base of type $\aleph_o$.

B. <u>THEOREM.</u> Let G be a subspace of βX which satisfies the

condition $1_{m,\aleph_0}$. Let E be a discrete subspace of G such that $|E| = m$ and $c\ell_{\mathcal{B}X} E = \mathcal{B}E$ and suppose that $\mathcal{F}$ is a filter base on E of type m. Let $\mu E = \{x \in \mathcal{B}E$: For every $Q \subset E$, if x is in $c\ell_{\mathcal{B}E} Q$, then $|Q| = m\}$. Then $|\mu E \cap ad_G \mathcal{F}| = 2^{2^m}$.

Proof. [J.E. VAUGHAN - (19) - Page 184]

C. THEOREM. Let S be an initially m-compact T_2-Space which contains a subspace Y such that $|Y| = m$ and $|c\ell_S Y| = 2^{2^m}$. Then there is a set B such that $Y \subset B \subset S$, $|B| \le 2^m$ and B is m-$\aleph_0$ compact.

Proof. [Victor Saks - (11) - Page 285]

D. LEMMA. The property $1_{m,n}$ is closed hereditary.

Proof. This is trivial from the fact that $K \cap G$ is compact for compact K and a closed set G.

We shall now give the example to show that $1_{m,n}$ is strictly stronger than m-n compactness.

Let Y be a discrete space of cardinality m. Let $S = \{0,1\}^{2^m}$, then there is a homeomorphism from $\mathcal{B}Y$ into the product space S. Hence $c\ell_{\mathcal{B}Y} Y = \mathcal{B}Y$ is contained in $c\ell_S Y$ and therefore $|c\ell_S Y| \ge 2^{2^m}$ but since $|S| = 2^{2^m}$, we have $|c\ell_S Y| = 2^{2^m}$. By Theorem C, there exists a set B such that $Y \subset B \subset S$, B is m-$\aleph_0$ compact and $|B| \le 2^m$. We shall prove that B does not satisfy the property $1_{m,\aleph_0}$. Let $G = B \cap \mathcal{B}Y$, then G is a closed subset of B which contained in $\mathcal{B}Y$ and contains Y. Let $\mathcal{F} = \{Y\}$, then $\mathcal{F}$ is a Filter base of type m but $|ad_G \mathcal{F}| \le |G| \le |\mathcal{B}| \le 2^m$. Therefore by Theorem B, G does not

satisfy the property $1_{m, \aleph_o}$ and hence by Lemma D, G does not satisfy the property $1_{m, \aleph_o}$.

7.2. A Space Which is $\aleph_o - \aleph_o$ Compact but not 'Strongly $\aleph_o - \aleph_o$ Compact'.

A. THEOREM. There exists a $\aleph_o - \aleph_o$ compact space P such that $\mathbb{N} \subset P \subset \beta\mathbb{N}$ with $|P| \leq 2^{\aleph_o}$, where $\mathbb{N}$ is the set of all positive integers.

Proof. [Frolik and others - (10)]

We shall see that the space which is of interest is P.

B. LEMMA. Every countable set in $\beta\mathbb{N}$ is C^*-embedded.

Proof. Let E be a countable subset of $\beta\mathbb{N}$ then we note that $E \cup \mathbb{N}$ is normal in $\beta\mathbb{N}$ and since E is closed in $E \cup \mathbb{N}$, E is C^*-embedded in $E \cup \mathbb{N}$. Hence E is C^*-embedded in $\beta\mathbb{N}$. (Since $\mathbb{N} \subset E \cup \mathbb{N} \subset \beta\mathbb{N}$ and $\mathbb{N}$ is dense in $E \cup \mathbb{N}$)

C. COROLLARY. Every infinite closed set in $\beta\mathbb{N}$ has cardinality $2^{2^{\aleph_o}}$.

Proof. Let G be an infinite closed subset of $\beta\mathbb{N}$, then G contains a copy of $\mathbb{N}$ and hence G contains a copy of $\beta\mathbb{N}$, which gives $|G| = |\beta\mathbb{N}| = 2^{2^{\aleph_o}}$.

D. Example. We note that the space P in the Theorem A does not contain infinite compact sets. Hence P is not in the class $\mathcal{C}^*$ which proves our necesscity. Therefore P is the space which satisfies the title of the sub-section.

8. Remarks.

This section is devoted to a study of the different definitions available for m-n compactness in the literature and their relations to the one given in this chapter.

8.1. The Variations of the Terminology.

A. Definition. A partially ordered set S of X is said to be $<n$-directed if each of its subsets of cardinality strictly less than n has an upper bound in S.

The term $\leq n$-directed is defined similarly. In this terminology a directed set is $< \aleph_o$ -directed.

Example. Let T be any set and let n be a regular cardinal. Then $S_n(T) = \{H \quad T : |H| < n\}$ is a $<n$-directed set with respect to the set inclusion.

Notation. We shall identify n with the smallest ordinal of cardinality n. Then $S_n(m)$ denotes the set of all subsets of m with cardinality less than n.

B. Definition. A m-n net in a set X is a function $\phi : S \to X$ where S is $<n$-directed and $|S| \leq m$.

C. Definition. A m-n sequence is a set in X whose domain is $S_n(m')$ for some m' where $n \leq m' \leq m$.

D. Definition. Let X be a topological space.

(i) X is said to be m-n compact (G) if and only if every m-n set in X has a cluster point.

(ii) X is said to be m-n compact $(\mathbb{N})$ if and only if every m-n sequence in X has a cluster point.

(iii) X is said to be m-n compact (S) if and only if every open cover of X of regular cardinality less than or equal to m has a sub-cover of cardinality less than n.

(iv) X is said to be m-n compact (A) if and only if for every open cover $\mathcal{U}$ of X with $|\mathcal{U}|$ regular and $n \leq |\mathcal{U}| \leq m$. There is a sub-cover $\mathcal{U}'$ of $\mathcal{U}$ such that $|\mathcal{U}'| < |\mathcal{U}|$.

Key. G - I.S. Gal

N - N. Noble

S - Smirnov

A - Alexandrov and Uryshon

8.2. The Connection Between (i),(ii),(iii),(iv) and The One We Used [JEV - (21)].

A. Definition. A point X in a topological space X is called a complete accumulation point of a set $S \subset X$ if for every neighbourhood U of x we have $|U \cap S| = |S|$.

B. THEOREM. The following are equivalent for a topological space X:

(i) X is m-n compact (A),

(ii) Let E be an infinite subset of X with cardinality α where α is regular and $n \leq \alpha \leq m$. Then E has a complete accumulation point.

(iii) Let $A_0 \supset A_1 \supset \cdots \supset A_\alpha \supset \cdots \supset, \alpha < d$ be a decreasing sequence of non-empty closed sets, well-ordered by inclusion with d regular and $n \leq d \leq m$. Then $\{A_i\}$ has a non-empty intersection.

[This is due to Alexandrov and Urysohn - Thro' (20).]

C. <u>THEOREM.</u> Let X be a topological space then we have the following:

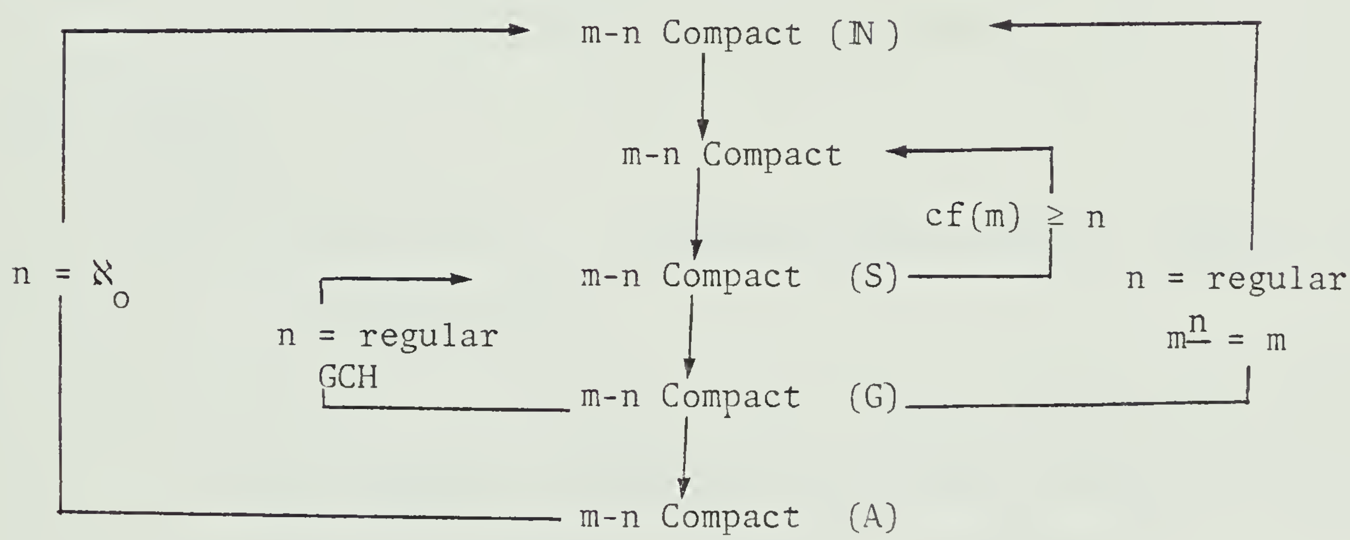
(i) X is m-n compact (G) if and only if every m-n stable filter base has an adherent point,

(ii) X is m-n compact $(\mathbb{N})$ if and only if for every m-n filter base $\mathcal{F}$, $\bigcap\{\mathcal{F}' : \mathcal{F}' \in S_\alpha(\mathcal{F})\} \neq \Phi$, for $n \leq \alpha \leq m$.

(iii) X is m-n compact (S) if and only if for every m-n filter base $\mathcal{F}$, with $|\mathcal{F}|$ regular, $\bigcap \overline{\mathcal{F}} \neq \Phi$.

[J.E. Vaughan - (21)]

D. <u>A Diagramme.</u>



<u>Note.</u> (i) The above diagramme is a reduction of <u>B</u> and <u>C</u>.

(ii) Theorem <u>C</u> shows these variations of the concept of m-n compactness also have m-n filter characterizations.

(iii) For $n = \aleph_0$, the $(\mathbb{N})$, (S), (G), (A) coincide with the Vaughan's definition.

(iv) For n-compactness Vaughan's definition coincides with $(\mathbb{N})$, (S) and (G) for regular n.

9. Main Spaces Considered in this Chapter.

We shall state the main spaces which are studied in this chapter with their relationship under certain conditions on the cardinals m and n. Let m and n be infinite cardinals such that n is regular and $m^{\underline{n}} = m$. Then we have the following:

$$
\begin{tikzcd}
\text{Compact} \arrow[dr] & & \\
& m\text{-Bounded} \arrow[d] \arrow[ddr] & \\
& \text{Strongly } m\text{-}n \text{ Compact} \arrow[d] & \\
& \text{The Property } 1_{m,n} \arrow[d] & m\text{-}n \text{ Bounded} \arrow[dl] \\
& m\text{-}n \text{ Compact} &
\end{tikzcd}
$$

10. Notes.

1. The theory of m-n compactness dates back to 1920's Alexandrov's and Uryshon's work.

2. The main three concepts in the Theory of Compactness (Compactness, Countable Compactness and Lindelöf Property) are special cases of the general concept m-n compactness.

3. We have seen that the partial solutions to the problems about productivity of compactness-like properties have been obtained through the general concept m-n compactness and some theorems about products have given in partial generality.

4. We wish to know, whether an arbitrary product of initially m-compact spaces is initially m-compact for regular cardinal number m and the possibility of deleting the condition

$m^{\underline{n}} = m$ from the hypothesis of the Theorem 3.3 (A).

The H(i) property and the generalized notion weak m-n compactness are similar to m-n compactness and we wish to mention that productivity of weak m-n compactness is a current research area.

In the next chapter we shall give some applications of maximal filters to compactness-like properties.

CHAPTER III

TOPICS OF INTEREST

1. Maximal Filters.

In this section we study basic properties of Maximal Filters on a discrete space X by paying more attention to large cardinals which will be useful in the next section to prove the existence of a strongly α-compact space which is not α-Bounded.

1.1. Uniform Ultrafilters.

A. Definition. Let X be a discrete space with cardinality α. Let Σ be an ultrafilter on X, then $\|\Sigma\|$ = minimum of $\{|G| : G \in \Sigma\}$ and if $\|\Sigma\| \geq k$, we say that Σ is k-uniform on X.

B. Definition. An ultrafilter Σ is said to be uniform on X if and only if $\|\Sigma\| = |X|$.

C. Notation. $U_k(X)$ = set of all k-uniform ultrafilters on X where $|X| \geq k$.

D. LEMMA. Let Σ be an ultrafilter on X. Then Σ is free if and only if $\|\Sigma\|$ is infinite.

Proof. '$\Longrightarrow$' Suppose $\|\Sigma\|$ is finite, then there exists a $G \in \Sigma$ such that $|G|$ is finite. Hence $G \cap (\cap \Sigma) \neq \Phi$.

'$\Leftarrow$' Suppose $\omega \in \cap\Sigma$, then $\{\omega\} \in \Sigma$. Hence $\|\Sigma\|$ is finite.

Note. Let $P^k(X) = \{S \in P(X) : |X-S| < k\}$ where $|X| = \alpha \geq k$ and k is infinite. Then any ultrafilter which contains $P^k(X)$ is k-uniform and by the lemma D, they are free.

1.2. Generalized Properties.

A. Definition. An ultrafilter Σ on X is said to be k-stable if and only if for every $\Sigma' \subset \Sigma$ with $|\Sigma'| < k$, $\cap\Sigma' \in \Sigma$.

B. LEMMA. Let Σ be an ultrafilter on X and suppose Σ is not k-stable. Then there exists a $\{G_i : i \in I\} \subset \Sigma$ where $|I| < k$ and $\bigcap_{i \in I} G_i = \Phi$.

Proof. Let k' be the smallest cardinal for which there is a family $\{G_i : i \in I\}$ where $|I| < k' < k$ and $\bigcap_{i \in I} G_i \notin \Sigma$. Then we define $\tilde{G}_o = X$ and $\tilde{G}_i = \bigcap_{\ell < i} G_\ell - (\bigcap_{i \in I} G_i)$ and we note that $\bigcap_{i \in I} \tilde{G}_i = \Phi$ $\tilde{G}_{i_2} \subset \tilde{G}_{i_1}$ for $i_1 < i_2 < k'$ and $\tilde{G}_i \in \Sigma$ for all $i \in I$.

We shall use this property in the next section.

C. Definition. Let $\Sigma \subset P(X)$. Then Σ is said to have k-uniform finite intersection property if $\Sigma \neq \Phi$ and $|\cap\Sigma'| \geq k$ for all $\Sigma' \subset \Sigma$ and $|\Sigma'|$ is finite.

Every k-uniform ultrafilter on X has the k-uniform finite intersection property.

D. LEMMA.

(i) Let $\Sigma \subset P(X)$ and $\Sigma \neq \Phi$. Suppose Σ has k-uniform finite intersection property, then $\Sigma \cup P^k(X)$ is a filter sub-base on X.

(ii) Let $|X| \geq k$ and k is infinite and let Σ be an ultrafilter on X. Then Σ is k-uniform if and only if $P^k(X) \subset \Sigma$.

(iii) The set $U_k(X) \subseteq \beta X - X$ is non-empty.

(iv) Every $\Sigma \subset P(X)$ with k-uniform finite intersection property can be extended to a member of $U_k(X)$.

Proof.

(i) Let $\Sigma_1 \subset \Sigma$ and $\Sigma_2 \subset P^k(X)$ with $|\Sigma_1|$ and $|\Sigma_2|$ are finite. Then $|\cap\Sigma_1| \geq k$ and $|X - \cap\Sigma_2| < k$ and hence we have $(\Sigma_1) \cap (\Sigma_2) \neq \Phi$.

(ii) '$\Longrightarrow$' Let $G \in P^k(X)$, then $|X-G| < k$ and therefore $X-G \notin \Sigma$.

'$\Longleftarrow$' Let $G \in \Sigma$ and suppose $|G| < k$. Then $X-G \in P^k(X) \subset \Sigma$ and therefore we have a contradiction.

(iii) We note that $P^k(X)$ has finite intersection property and hence $P^k(X)$ is contained in some ultrafilter.

(iv) Follows from (i).

1.3. Convergence.

A. Definition. An ultrafilter Σ is said to be convergent at $x \in X$ if and only if the neighbourhood filter at x, V_x is contained in Σ.

B. <u>LEMMA.</u> Let X be discrete space and let $x \in \beta X$. Suppose Σ_x is an ultrafilter such that $\Sigma_x \to x$, then $\{C\ell_X G : G \in \Sigma_x\}$ is a neighbourhood base at x.

<u>Proof.</u> Let U be an open neighbourhood of x in βX. Then $\beta X\text{-}U$ is a closed subset of βX and $x \notin \beta X\text{-}U$ and by the definition of the base for βX, there exists a $B \subset X$ such that $\beta X - U \subset C\ell_{\beta X} B$ and $x \notin C\ell_{\beta X} B$. Hence $x \in C\ell_{\beta X}(X - B) \subset U$ and it is easy to see that $X - B \in \Sigma_x$ and $C\ell_{\beta X}(X\text{-}B)$ is open.

C. <u>Notation.</u> We denote $\hat{A}_k = C\ell_{\beta X} A \cap U_k(X)$ considering $U_k(X)$ as a subspace of βX.

D. <u>LEMMA.</u> Let $|X| \geq k$ and k is infinite. Then we have the following:

(i) Let $A, B \in P(X)$, then $(A \hat{\cap} B)_k = \hat{A}_k \cap \hat{B}_k$.

(ii) $U_k(X)$ is a compact subspace.

(iii) Let $A \in P(X)$, then $|A| < k$ if and only if $\hat{A}_k = \Phi$.

(iv) Let $A, B \in P(X)$, then $\hat{A}_k \subset \hat{B}_k$ if and only if $|A\text{-}B| < k$.

<u>Proof.</u>

(i) We note that, since X is a discrete space, $C\ell_{\beta X}(A \cap B) = C\ell_{\beta X} A \cap C\ell_{\beta X} B$.

(ii) Let $x \notin U_k(X)$, then the corresponding ultrafilter $\Sigma_x \to x$ and $\Sigma_x \notin U_k(X)$. Hence by lemma <u>B</u> and the definition of k-uniform ultrafilters there exists a $G \in \Sigma_x$ such that $|G| < k$ and $x \in C\ell_{\beta X} G$ (open

neighbourhood of x) and $C\ell_{\beta X} G \cap U_k(X) = \Phi$.

Therefore $U_k(X)$ is compact.

(iii) '$\Longrightarrow$' Suppose $|A| < k$ and let $x \in C\ell_{\beta X}A$. Then $A \in \Sigma_x$ and hence $\|\Sigma_x\| < k$. Therefore x is not in $U_k(X)$ and hence $C\ell_{\beta X}A \cap U_k(X) = \Phi$.

'$\Longleftarrow$' Suppose $|A| \geq k$, then by lemma 1.2-D, $U_k(A) \neq \Phi$. Let $\Sigma_x \in U_k(A) \subset U_k(X)$ and hence $x \in C\ell_{\beta X}A$ and $x \in U_k(X)$. Therefore $\hat{A}_k \neq \Phi$.

(iv) '$\Longrightarrow$' Suppose $|A-B| \geq k$, then by (iii) $x \in \widehat{(A-B)}_k \subset \hat{A}_k \subset \hat{B}_k$ and hence $B \in \Sigma_x$ but $(A-B) \in \Sigma_x$. Therefore we have the implication to the right.

'$\Longleftarrow$' Since $|A-B| < k$, we know $\widehat{(A-B)}_k$ is empty and therefore $\hat{A}_k = \widehat{(A \cap B)}_k$ and hence $\hat{A}_k \subset \hat{B}_k$.

1.4. Selective Ultrafilters.

A. Definition. Let α be infinite and $x \in U_\alpha(X)$ where $|X| = \alpha$. Then an ultrafilter Σ_x on X is said to be selective if and only if for every partition $\{X_i : i \in I\}$ of X where $|I| < \alpha$, either there is an $i \in I$ such that $X_i \in \Sigma_x$ or there is a $G \in \Sigma_x$ such that $|G \cap X_i| \leq 1$ for all $i \in I$.

B. LEMMA. Let $\{X_i : i \in I\}$ be a partition of X where $|I| < \alpha$. Then $\Sigma_x \in U_\alpha(X)$ is selective if and only if there is a $G \in \Sigma_x$ such that $|\{i \in I : |G \cap X_i| > 1\}| \leq 1$.

Proof. '$\Longrightarrow$' Let $\Sigma_x \in U_\alpha(X)$ be a selective ultrafilter, then if $X_i \in \Sigma_x$ the result is trivial otherwise there is a $G \in \Sigma_x$

which satisfies the requirement (Trivial too).

'$\Leftarrow$' Suppose $|G \cap X_{i'}| > 1$ for some $i = i'$ and $|G \cap X_i| \leq 1$ for all $i \neq i'$. We note that

$G = (\bigcup_{i \neq i'} (G \cap X_i)) \cup (G \cap X_i')$ and since $G \in \Sigma_X$ either $G \cap X_{i'} \in \Sigma_X$ or $\bigcup_{i \neq i'} (G \cap X_i') \in \Sigma_X$ but $|\bigcup_{i \neq i'} (G \cap X_i)| = \sum_{i \neq i'} |G \cap X_i| \leq |I| < \alpha$. Hence $G \cap X_{i'} \in \Sigma_X$ and therefore $X_{i'} \in \Sigma_X$.

C. Definition. Let X be an uncountable discrete space with $|X| = \alpha$. Then α is said to be measurable if and only if there is an α-stable free ultrafilter on X.

In Chapter I, we have defined measurable cardinals and we note that the above definition is convenient to work with in this Chapter and also it is easy to see that the new definition is equivalent to the old one.

D. THEOREM. Let α be a measurable cardinal, then $U_\alpha(X)$ contains a selective ultrafilter where $|X| = \alpha$.

Proof. See [(3) - Page 212].

2. Measurable Cardinals.

In this section we wish to give a space which is strongly α-compact but not α-Bounded, under the assumption of existence of a measurable cardinal α. This construction is due to JEV (22) and we believe that example is hard but for the completion we outline the method.

2.1. P_k-Points.

A. Definition. Let k be infinite and let $x \in X$, then x is said to be a P_k-point of X if and only if x has a $<$ k-stable neighbourhood base.

We note every point of a topological space is $P_{\aleph_0}$ and $P_{\aleph_1}$-points are called P-points, in $T_{3\frac{1}{2}}$-spaces.

B. LEMMA. Let α be uncountable and $x \in U_\alpha(X)$ where X is a discrete space with $|X| = \alpha$. Then $\{\hat{G}_\alpha : G \in \Sigma_x\}$ is an open neighbourhood base at x in $U_\alpha(X)$.

Proof. We note that $\{C\ell_X G : G \in \Sigma_x\}$ is an open neighbourhood base at x in X and hence the lemma.

C. THEOREM. Let X be a discrete space with $|X| = \alpha$. Let α and k be infinite. Suppose $x \in X$ is a P_k-point of $U_\alpha(X)$. Then we have $k > Cf(\alpha)$ or Σ_x is k-stable.

Proof. Suppose Σ_x is not k-stable and $k \leq Cf(\alpha)$. Then there exists a $\{G_i : i \in I\}$ contained in Σ_x where $|I| < k$ and $\bigcap_{i \in I} G_i = \Phi$ (1.2-B). Since x is a P_k-point of $U_\alpha(X)$, there exists a $G \in \Sigma_x$ such that $\hat{G} \subset \bigcap_{i \in I} \hat{G}_i$ and hence $\hat{G} \subset \hat{G}_i$ for all $i \in I$. We note that $|G-G_i| < \alpha$ and $G = \bigcup_{i \in I} (G-G_i)$. By hypothesis, $|I| < Cf(\alpha)$ and hence $|G| < \alpha$. Therefore $\|\Sigma_x\| < \alpha$ and hence we have a contradiction which gives the theorem.

D. THEOREM. Let X be a discrete space with $|X| = \alpha$. Let

α be infinite and $\Sigma_x \in U_\alpha(X)$. Suppose Σ_x is selective, then x is a P_α-point.

Proof. Let $\{G_\gamma : \gamma \in I\} \subset \Sigma_x$ where $G_o = X$ and $|I| < \alpha$. Let $G = \bigcap_{\gamma \, I} G_\gamma$. Suppose $G \in \Sigma_x$, then x $\hat{G} \subset G_\gamma$ for all $\gamma \in I$. Suppose $G \notin \Sigma_x$, then Σ_x is not α-stable and by lemma 1.2-B, we can take $G = \Phi$. Let $X_o = X - G_o$ and $X_\beta = (\bigcap_{\gamma<\beta} G_\gamma) - G$. Then we note that $X_{\gamma_1} \cap X_{\gamma_2} = \Phi$ for $\gamma_1 \neq \gamma_2$ and $\bigcup_{\gamma\leq\beta} X_\gamma = X - \bigcap_{\gamma\leq\beta} G_\gamma$. Hence $\{X_\gamma : \gamma \in I\}$ forms a partition for X and since Σ_x is selective and $X_\beta \notin \Sigma_x$ for $0 < \beta < \alpha$, there exists a $H \in \Sigma_x$ such that $|H \cap X_\beta| \leq 1$ for all $\beta < \alpha$. We note that $H \cap (\bigcup_{\gamma\leq\beta} X_\gamma) \supseteq H - G_\beta$ and hence $|H-G_\beta| \leq \sum_{\gamma\leq\beta} |H \cap X_\gamma| < \alpha$ for all $\beta < \alpha$. By lemma 1.3-D $\hat{H} \subset \hat{G}_\beta$ for all $\beta < \alpha$ and hence $\hat{H} \subset \bigcap_{\gamma \in I} \hat{G}_\gamma$. Theorem follows from lemma 2.1-B.

2.2. Example.

A. LEMMA. Let X be a discrete space with $|X| = \alpha$. Let α be measurable, then there exists a α-stable, selective, free ultrafilter Σ_x in $U_\alpha(X)$.

Proof. By 1.4-D, $U_\alpha(X)$ contains a selective ultrafilter Σ_x (say) and by the previous lemma, x is a P_α-point. It is clear that Σ_x is free and we note that α is strongly inaccessible, see (Chapter I, 1.4-D) and hence $\alpha = Cf(\alpha)$. Therefore by 2.1-C Σ_x is α-stable.

B. LEMMA. Let Y be a compact T_2-space having a dense subset of cardinality α and let x be a non-isolated point of Y such that no filter base σ on $Y-\{x\}$ with $|\sigma| \leq \alpha$ converges to x in Y. Then the subspace $Y-\{x\}$ is strongly α-compact but not α-bounded.

Proof. See [JEV-(22), Page 360].

C. LEMMA. Let Σ_x be a α-stable, selective free ultrafilter in $U_\alpha(X)$ where X is a discrete space with cardinality α. Then there is no filter base σ on $\beta X-\{x\}$ with $|\sigma| \leq \alpha$ and $\sigma \to x$ in X.

Proof. See [JEV-(22), Page 360].

D. Method. Let X be a discrete space with $|X| = \alpha$. Let $Y = \beta X$ and let α be a measurable cardinal. Then we note that every point of $\beta X-X$ is non-isolated and there exists a α-stable, selective, free ultrafilter Σ_x in $U_\alpha(X)$. By lemma C and lemma B, $\beta X-\{x\}$ is strongly α-compact but not α-bounded.

3. Γ-limit points.

The sequences were demand objects in the theory of convergence in topological spaces but a better object is now available, namely Filters (Ultrafilters). The concept of Γ-limit is defined in terms of ultrafilters Γ in $\beta N-N$. Using Γ-limits we can define the concept of Γ-compactness [see - (1)].

3.1. Fundamental Facts.

A. Definition. Let Γ be a non-principal ultrafilter on N.

Let X be a topological space and (x_n) be a sequence in X, then a point $x \in X$ is said to be a <u>Γ-limit point</u> of (x_n) if for every neighbourhood W of x, $\{n : x_n \in W\} \in \Gamma$.

We know that $\|\Gamma\|$ is countably infinite for every Γ in $\beta N - N$ and therefore if $x_n \to x$, then x is a Γ-limit point of (x_n) for every Γ in $\beta N - N$.

B. <u>Examples.</u>

(i) Let X be the set of all non-negative integers as a subspace of reals. Then we define $x_n = 1$, n = even

$= 0$, n = odd.

Let E be the set of all even positive integers. Then $P^{\aleph_0}(E)$ is a free filter base on N. Let Γ be an ultrafilter containing $P^{\aleph_0}(E)$ and we note that $\Gamma \in \beta N - N$ and 1 is a Γ-limit point of (x_n) but $x_n \not\to 1$.

We restrict ourselves to non-principal ultrafilters on N to avoid trivialities.

(ii) Let X be an infinite set with particular point topology $\tau_{x_0} = \{U : x_0 \in U\} \cup \{\Phi\}$ for some $x_0 \in X$.

We note that for every $x \in X$, there exists an open set W such that $x \in W$ and $|W| = 2$. Therefore any sequence (x_n) in X with distinct elements has no Γ-limit points.

It is easy to see that in general Γ-limits are not unique and trivially non-principal ultrafilters are

dominating on N, which is the case always for Z-ultrafilters.

C. THEOREM.

(i) Let $f : x \to y$ be a continuous function and let (x_n) be a sequence in X and let $x \in X$ be a Γ-limit point of (x_n). Then $f(x)$ is a Γ-limit point of $(f(x_n))$.

(ii) Let x be a Γ-limit point of (x_n). Then (x_n) clusters at x.

Proof.

(i) We note that $\{n : f(x_n) \in W\} = \{n : x_n \in f^{-1}(W)\}$ for every $W \in V_{f(x)}$ and hence we have (i).

(ii) Suppose x is not a cluster point of (x_n). Then for given $n_o \in N$, there exists a W (neighbourhood of x) such that $|\{n : x_n \in W\}| \leq n_o$, hence x is not a Γ-limit point. Therefore we have the theorem.

D. THEOREM.

(i) Let (x_n) be a sequence in X and let x be a cluster point of (x_n). Then there exists a in $\beta N - N$ such that x is a Γ-limit point of (x_n).

(ii) Let X be a T_2-space, then the Γ-limits in X are unique.

Proof.

(i) We denote V_x, the neighbourhood system at x and let $S(W) = \{n : x_n \in W\}$ where $W \in V_x$. Then

$\Sigma = \{S(W) - \{n\} : W \in \mathcal{V}_x$ and $n \in N\}$ is a filter sub-base on N. Let Γ be an ultrafilter containing Σ, then $\Gamma \in \beta N - N$ and since $S(W) \in \Gamma$ for every $W \in \mathcal{V}_x$, x is a Γ-limit point of (x_n).

(ii) Suppose (x_n) has two Γ-limit points in the space X, say x and y. If $x \neq y$, then there exist open sets U_x, U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \Phi$. Hence Γ will contain an empty set, which gives a contradiction.

Note. In (i) Γ-depends on the point x.

4. Γ-Compact Spaces.

In this section we shall give an introduction to Γ-compact spaces and also we shall prove an easy product theorem for Γ-compact spaces—(1).

4.1. Fundamental Facts.

A. Definition. Let Γ be a non-principal ultrafilter on N, then a topological space X is said to be Γ-compact if every sequence in X has a Γ-limit point.

A space X is said to be ultracompact if X is Γ-compact for every Γ in $\beta N - N$.

B. Proposition. Suppose X has the property that the closure of any countable set in X is compact, then X is ultracompact.

Proof. Suppose X is not ultracompact, then there exists a Γ in $\beta N - N$ such that X is not Γ-compact. Hence there exists a (x_n) in X such that (x_n) has no Γ-limit points. Let $x \in \overline{(x_n)}$, then there exists an open set U_x such that $x \in U_x$ and $\{n : x_n \in U_x\} \notin \Gamma$. Hence $\{n : x_n \notin U_x\} \in \Gamma$. Since $\overline{(x_n)}$ is compact, there exist $x^{(1)}, x^{(2)}, x^{(3)} \ldots x^{(n)}$ in $\overline{(x_n)}$, such that

$\overline{(x_n)} \subseteq \bigcup_{i=1}^{n} U_x(i)$ and hence $\bigcap_{i=1}^{n} \{n : x_n \notin U_x(i)\} = \Phi$. Therefore we have a contradiction and hence X is ultracompact.

C. THEOREM.

(i) The continuous image of a Γ-compact space is Γ-compact.

(ii) Every closed subspace of a Γ-compact space is Γ-compact.

Proof.

(i) Let $f : x \rightarrow y$ be a continuous, onto function. Suppose X is Γ-compact and let (y_n) be a sequence in y. Then, since f is onto, there exist x_n for each n such that $y_n = f(x_n)$. Let x be a Γ-limit of (x_n), then by theorem 3.1-C, $f(x)$ is a Γ-limit point of (y_n) and hence Y is Γ-compact.

(ii) Let H be a closed subspace of a Γ-compact space X. Let (x_n) be a sequence in H, then (x_n) has a Γ-limit point x (say) in X and it is easy to see that $x \in H$. Hence every sequence in H has a Γ-limit point in H. Therefore H is Γ-compact.

(We shall prove that arbitrary products of Γ-compact spaces is Γ-compact.)

D. <u>THEOREM.</u> Let $X = \prod_{i\ I} X_i$ where each X_i is non-empty. Then X is Γ-compact for some Γ in $\beta N - N$ (or βN) if and only if each X_i is Γ-compact.

<u>Proof.</u> '$\Longrightarrow$' Follows from the fact that each π_i is continuous and onto.

'$\Longleftarrow$' Let (x_n) be a sequence in X, then for each $i \in I$, $(x_{n,i})$ is a sequence in X_i and since X_i is Γ-compact, $(x_{n,i})$ has a Γ-limit point x_i (say) in X_i. Let $x = (x_i)$ and let U be a basic open set in X_i such that $x \in U$. Suppose the non-trivial factors of U are U_{i_ℓ}, $\ell = 1, 2, \ldots n$, then

$\{n : x_{n,i_\ell} \in U_{i_\ell}\} \in \Gamma$; $\ell = 1, 2, \ldots n$ and hence we note that

$\{n : x_n \in U\} \supseteq \bigcap_{\ell=1}^{n} \{n : x_{n,i_\ell} \in U_{i_\ell}\}$. Therefore x is a Γ-limit point of (x_n) in the product space X and hence we have the theorem.

1. <u>Notes.</u>

$$\begin{tikzcd}
\text{Compact} \arrow[r] & \text{Ultracompact} \arrow[d] \\
 & \Gamma\text{-compact for some } \Gamma \text{ in } \beta N - N \arrow[d] \\
 & \text{Countably compact} \arrow[d] \\
 & \text{Limit-point compact}
\end{tikzcd}$$

2. Well-known spaces.

(i) compact spaces

(ii) countably compact spaces

(iii) Lindelöf spaces

(iv) H(i)-spaces

3. Spaces of interest.

(i) weakly Lindelöf spaces

(ii) Γ-compact spaces

(iii) ultracompact spaces

(iv) feebly compact spaces and weakly Γ-compact spaces.

We shall give some basic but important examples in the study of compactness-like properties.

List of Examples

1. Particular point topology on an infinite set is an trivial example but it is important. Let X be an infinite set and let τ_d be the particular point topology on X. Then (X,τ_d) is

   (i) not-compact,

   (ii) not Γ-compact but

   (iii) H(i)

   We see that $A = \{d\}$ is compact and $\bar{A} = X$ is not compact. By selecting the cardinality of X properly, the above space can be taken as an example for a weakly m-n compact space which is not m-n compact. [See Chapter I-3.1]

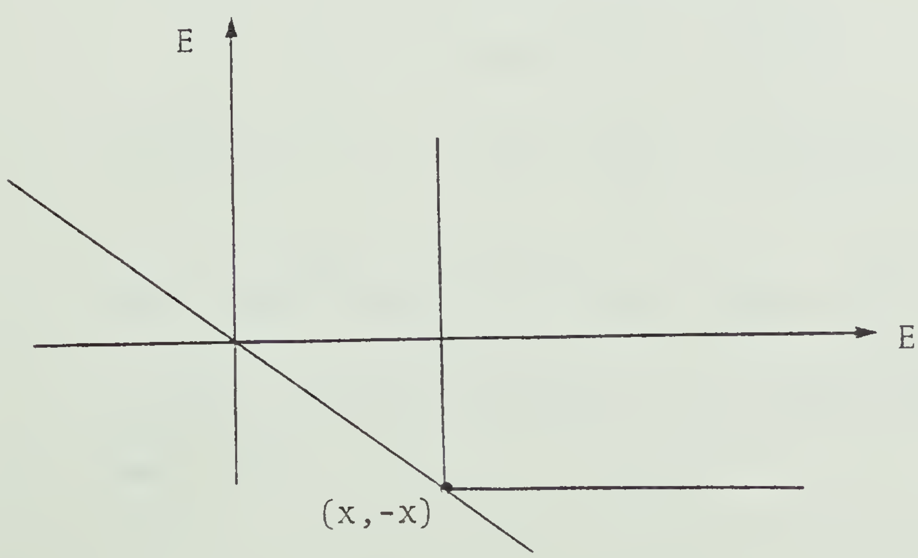
2. One of the basic examples in general Topology is the Sorgenfry line E. ($\mathbb{R}$ with right open intervals as basic open sets). The space E has the following properties:

   (i) Lindelöf, non-compact,

   (ii) T_3 ,

   (iii) Paracompact,

   (iv) $E \times E$ is not Lindelöf.



3. Let $S = I \times I$ with dictionary order topology. Then we have the following where I is the unit interval:

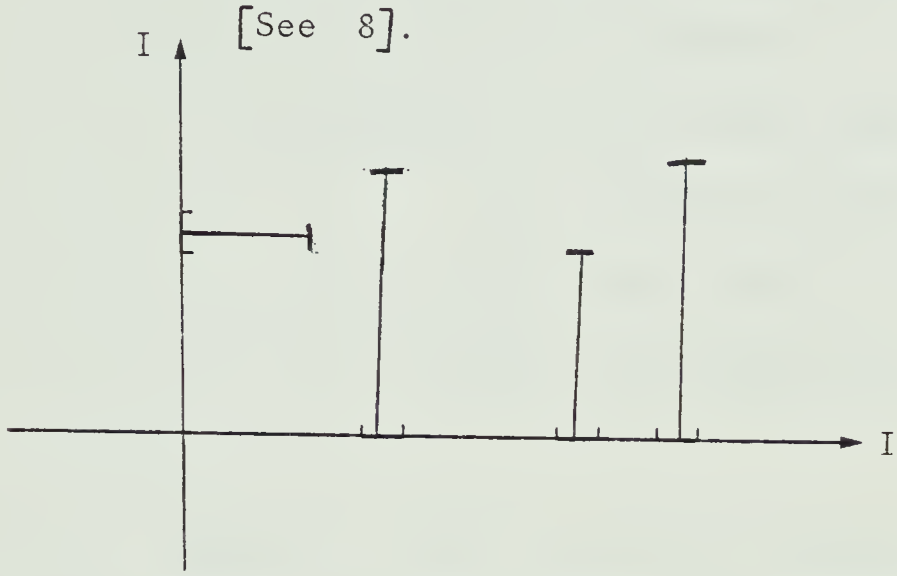
(i) S^+ (S with basic open sets $[x,y)$) is Lindelöf ,

(ii) S^- (S with basic open sets $(x,y]$) is Lindelöf ,

(iii) $d(S) = \aleph_1$,

(iv) $S^+ \times S^-$ is not weakly Lindelöf.

[See 8].



4. Let Ω be the set of all ordinals less than or equal to the first uncountable ordinal ω_1 . Let $\Omega_o = \Omega - \{\omega_1\}$, then every countable subset A of Ω_o has an upper bound in Ω_o . We note the following:

(i) Ω is compact,

(ii) Ω_o is non-compact,

(iii) Ω_o is first countable,

(iv) Ω_o is sequentially compact.

We know that Ω_o is an open subspace of Ω .

5. Let $T = \mathbb{N}^A$ where $|A| \geq \aleph_1$, then $\mathbb{N}^A$ is not normal. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, then $S \subset I$ and is homeomorphic to $\mathbb{N}$.
See [14] .

6. The space I^I where $I = [0,1]$.

(i) I^I is compact and Hausdorff,

(ii) I^I contains a non-normal space S^I ,

(iii) I^I is not sequentially compact for (Let $\alpha_n : I \to I$, and $\alpha_n(x)$ = nth digit in the binary expansion of x . Then $\{\alpha_n\}$ has no convergent subsequence).

(iv) I^I is not first countable.

In addition to this we note that $\Omega_o \times I^I$ is,

(i) countably compact,

(ii) not compact,

(iii) not sequentially compact.

7. (i) Examples of spaces which are countably compact but not Γ-compact.

(ii) Examples of spaces which are ultra-compact but not compact.

See [1]

8. Let X be an infinite discrete space with cardinality m . Let $S = \{0,1\}^{2^m}$, then there is a homeomorphism from $\mathscr{B}X$ into $\{0,1\}^{2^m}$ and a set B such that $X \subset B \subset S$. The set B is intially m-compact but does not satisfy $1_{m,\aleph_o}$. See [Chapter II-7.1]

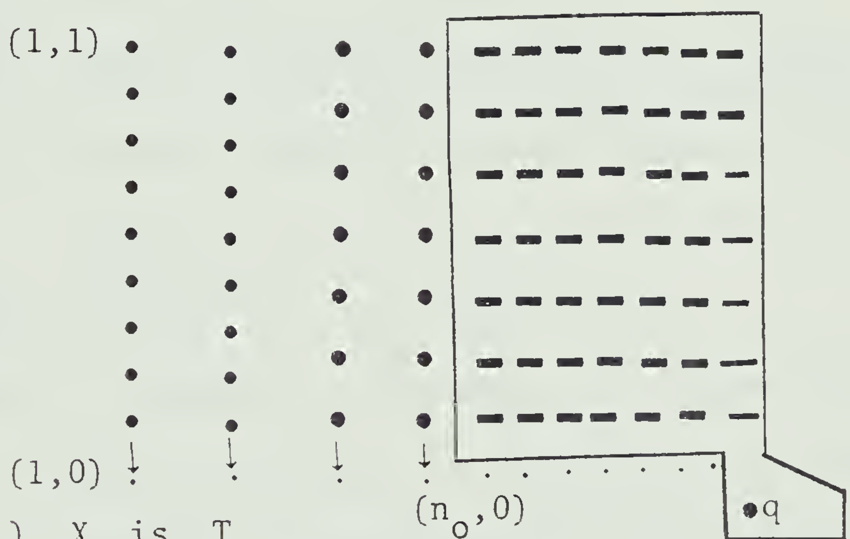
9. Let $X = \mathbb{R}$, then the neighbourhood system $\mathscr{V}_x$ at x is a fixed m-n filter. See [Chapter II-2.1]

10. Let $|X| = m$ and $S \subset X$ with $|S| = n$ where n is regular. Then $P^{\aleph_o}(S)$ is a free m-n filter filter base. See [Chapter II-2.1]

11. Most of the <u>hard</u> examples in our work are based on subspaces of the Stone-Cêch compactification of a discrete space X . The techniques and examples are studied in [19] .

Modern Applied Mathematics deal with abstract topological spaces with some additional structure. In most of the work in this nature we assume the T_2-property of the space. (Topological Manifolds).

12. Let $\mathbb{N}^* = \{1/n: \ n \in \mathbb{N}\} \cup \{0\}$ and let X be the space obtain by adjoining an ideal point 'q' to $\mathbb{N} \times \mathbb{N}^*$. The topology on X is determined by the product topology on $\mathbb{N} \times \mathbb{N}^*$ together with basic neighbourhoods $U_{n_o}(q) = \{q\} \cup \{(n,1/m): \ n > n_o \ , \ m \in \mathbb{N}\}$.



(i) X is T_2

(ii) The space $\{(n,0): \ n \in \mathbb{N}\}$ is a discrete closed subspace of X .

(iii) Let $S = \{1, 2, \ldots n_o\}$, then $S \times \mathbb{N}^*$ is compact

(iv) $\overline{U_{n_o}(q)} \cup (S \times N^*) = X$.

We see that X is $\mathscr{H}(i)$ but not compact.

13. Hilbert space is the set of all sequences (x_i) of real numbers such that $\sum x_i^2$ converges and the topology is generated by the metric $d(x,y) = (\sum(x_i-y_i)^2)^{\frac{1}{2}}$. This space is homeomorphic to $\mathbb{R}^\omega$, $|\omega| = \aleph_0$. The subspace I^ω of the Hilbert space is called the Hilbert cube. I^ω is clearly compact and has the following properties:

(i) I^ω is a metric space and hence it has all separation properties $(T_0, T_1, T_2, T_3 \ldots)$

(ii) I^ω is separable and hence it has all countability properties.

14. <u>Double pointed countable complement topology</u>. Let X be an uncountable set and the topology on X is generated by complements of countable sets and Φ.

(i) X is T_0, T_1 but not T_2

(ii) Since the <u>subspace topology on a countable set is discrete</u>, the only compact sets are finite and hence X is neither compact nor countably compact but it is Lindelöf.

(iii) Trivially X is $<\aleph_1$-discrete

(iv) A variation of the space X can be constructed by doubling each of its points. Technically this doubling is done by taking the usual product with an indiscrete space $Y = \{0,1\}$. Let $D = X \times Y$, then D has the property that every infinite set has a limit point. This property is called the <u>limit-point compactness</u>. The original space X does not have this property.

D is an example of a space which is <u>Limit-point compact</u> <u>but not $\aleph_0 - \aleph_0$ compact.</u>

15. Let X be the Cartesian product fo the real line with usual product topology and $\{0,1\}$ with the indiscrete topology. We shall show that intersection of two compact sets need not to be compact.

$$\text{Let } A_1 = [a,b] \times \{0\}$$
$$B_1 = (a,b) \times \{1\}$$
$$A_2 = [a,b] \times \{1\}$$
$$B_2 = (a,b) \times \{0\} .$$

We note that every open set that contains $(a,0)$ contains $(a,1)$ and hence $S_1 = A_1 \cup B_1$ and $S_2 = A_2 \cup B_2$ are compact but $S_1 \cap S_2 = (a,b) \times \{0,1\}$ is not compact.

We know that intersection of any collection of compact sets of a T_2-space is compact.

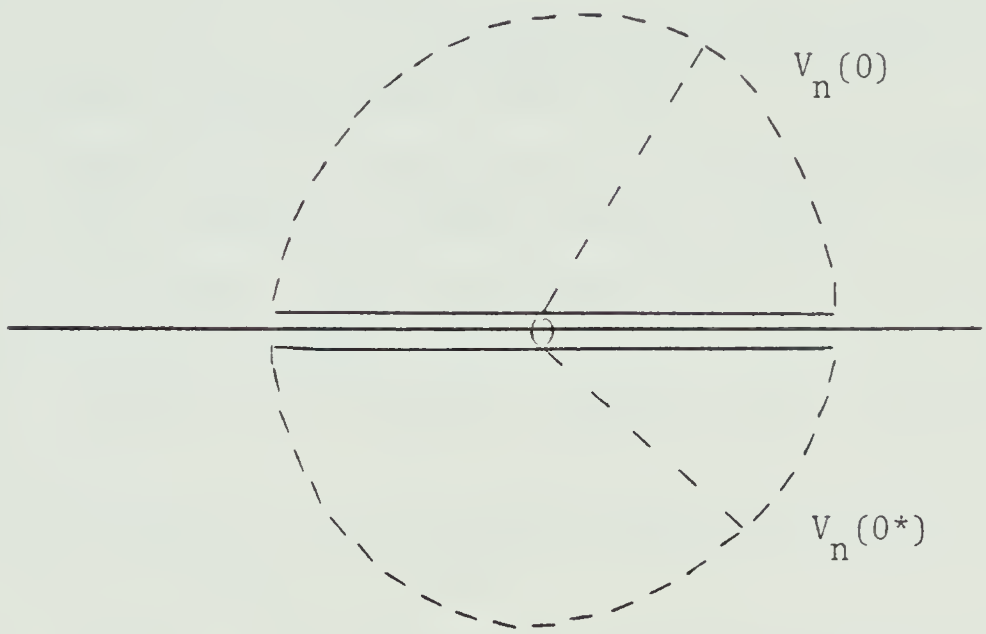
16. The space $\mathbb{N}$ is not countable compact but $\mathbb{N}$ is open in $\mathcal{L}\mathbb{N}$. Therefore Γ-compactness is not hereditary.

17. Let $\Omega_o = \Omega - \{\omega_1\}$. Then we know that closure of any countable set is compact and hence Ω_o is ultracompact but Ω_o is not a closed subset of Ω . Therefore Γ-compact subspaces behave differently to compact-subspaces in T_2-spaces. See [Chapter III-4.1].

18. Double Origin Topology. Let X consists of points of the plane $\mathbb{R}^2$ together with the additional point 0^* . The neighbourhoods of points other than 0 and 0^* are the usual open sets of $R^2 - \{0\}$. The neighbourhoods of 0 and 0^* are defined as follows:

$$V_n(0) = \{(x,y):\ x^2 + y^2 < 1/n^2 \quad y > 0\} \cup \{0\}$$

$$V_n(0^*) = \{(x,y):\ x^2 + y^2 < \frac{1}{n^2} \quad y < 0\} \cup \{0^*\}$$



This is T_2 but not $T_{2\frac{1}{2}}$ and hence not compact. This space is clearly second countable and hence it is Lindelöf.

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APPENDIX

WEAKLY m-n COMPACT SPACES

1. Fundamental Facts.

The Property H(i) and weakly-Lindelof are special cases of the general concept weak m-n compactness. This section is devoted to a study of basic properties of weakly m-n compact spaces. We shall begin the section by giving the basic definitions.

1.1 Basic Notions.

A. Definition. A topological space X is said to be weakly m-n compact if every m-fold open cover of X has a sub-family of cardinality strictly less than n with dense union.

B. Definition. A topological space X is said to be weakly n-compact if it is weakly m-n compact for each $m \geq n$.

C. Special Cases.

(i) Weakly $\aleph_0 - \aleph_0$ compact spaces $\equiv$ weakly countably compact spaces

(ii) Weakly $\aleph_0$-compact spaces $\equiv$ H (i) spaces.

(iii) Weakly $\aleph_1$-compact spaces $\equiv$ weakly-Lindelof spaces

(iv) Weakly m- $\aleph_0$ compact spaces $\equiv$ Initially weakly m-compact spaces.

Note. In a T_2-space H(i) is equivalent to H-closed.

D. Definition. A subset E of X is said to be weakly m-n compact

if every m-fold open cover $\mathcal{U}$ of E by open subsets of X has a sub-family $\mathcal{U}'$ (say) of cardinality strictly less than n and $E \subseteq \overline{\bigcup \mathcal{U}'}$

Note: A subset E of X is said to be weakly m-n compact if and only if E is weakly m-n compact with respect to its subspace topology.

1.2 Elementary Properties.

A. THEOREM. (i) Let n be a regular cardinal number. Then k-fold union of weakly m-n compact subspaces of a fixed space is weakly m-n compact for all $k < n$.

(ii) A space which contains a dense weakly m-n compact is weakly m-n compact.

Proof. (i) Let $\{A_i\}$ be a k-fold collection of weakly m-n compact subspaces of the space X. Let $\mathcal{U}$ be a m-fold open cover of $\bigcup_{i \in I} A_i$ where $|I| = k<n$. Then there exist sub-families of $\mathcal{U}$, $\{\mathcal{U}_i : i \in I\}$ (say) such that $A_i \subset \overline{\bigcup \mathcal{U}_i}$ and $|\mathcal{U}_i| < n$ for all $i \in I$. We note that $\bigcup_{i \in I} \overline{(\bigcup \mathcal{U}_i)} \subseteq \overline{\bigcup_{i \in I} (\bigcup \mathcal{U}_i)}$ and $|\bigcup_{i \in I} \mathcal{U}_i| < n$ and hence the (i).

(ii) Let A be a dense weakly m-n compact subspace of X. Let $\mathcal{U}$ be a m-fold open cover of X. Then, since A is weakly m-n compact there exists a sub-family $\mathcal{U}' \subset \mathcal{U}$ such that $A \subseteq c\ell_X(\bigcup \mathcal{U}')$ and $|\mathcal{U}'| < n$. Hence we have $X = \overline{\bigcup \mathcal{U}'}$ which proves (ii).

B. Definition. A subset A of the space X is said to be regular closed if and only if $A = c\ell_X \operatorname{Int}_X A$.

C. THEOREM. (i) The continuous image of a weakly m-n compact space is weakly m-n compact.

(ii) A regular closed subspace of a weakly m-n compact space is weakly m-n compact.

Proof. (i) Let $f : X \to Y$ continuous and onto. Suppose X is weakly m-n compact and let $\mathscr{U}'$ be an m-fold open cover of Y. Then $\{f^{-1}(U): U \in \mathscr{U}'\}$ is an m-fold open cover of X and hence there exists a sub-family $\mathscr{U}'$ of $\mathscr{U}$ such that $X = \overline{\bigcup\{f^{-1}(U)\ U \in \mathscr{U}'\}}$ and $|\mathscr{U}'| < n$. Hence we have $Y = \bigcup\{U: U \in \mathscr{U}'\}$ which proves (i).

(ii) Let H be a regular closed subset of the space X which is weakly m-n compact. Let $\mathscr{U}$ be a m-fold open cover of H by open subsets of X. Then $X = (\bigcup\mathscr{U}) \cup (X-H)$ and since X is weakly m-n compact, there exists a $\mathscr{U}' \subset \mathscr{U}$ such that $X = (\overline{\bigcup\mathscr{U}'}) \cup \overline{(X-H)}$ and $|\mathscr{U}'| < n$. Hence $\mathrm{Int}_X H \subseteq \overline{\bigcup\mathscr{U}'}$ and since $H = c\ell_X \mathrm{Int}_X H$, we have $H \subseteq \overline{\bigcup\mathscr{U}'}$. Therefore H is weakly m-n compact.

We note that the property weak m-n compactness is regular closed hereditary.

D. Example. Let X be a space with $|X| = m > n$ and particular point topology τ_d. Then $\tau_d = \{\mathscr{U}: d \in \mathscr{U}\} \cup \{\Phi\}$ where $d \in X$. We note that $X - \{d\}$ is closed and discrete and hence $X-\{d\}$ is not weakly m-n compact, but X is weakly m-n compact. This example shows that weak m-n compactness is not closed hereditary.

Note. The space (X,τ_d) is not m-n compact.

1.3 Special Properties.

A. Definition. A space X is said to be m-separable if X contains a dense subset of cardinality less than m.

In this terminology separable spaces are $\aleph_1$ - separable.

B. Note. Every n-separable space is weakly m-n compact.

C. LEMMA. Weak m-n compactness is a topological property.

Proof. Follows from the fact that continuous maps preserve weak m-n compactness.

D. Proposition Let n be a regular cardinal. Then the product of a weakly m-n compact space with a n-separable space is weakly m-n compact.

Proof. Let X be a weakly m-n compact space and let Y be a n-separable space. Then Y contains a dense subset A with $|A|<n$. We note that $X \times A$ is weakly m-n compact and since $\overline{X \times A} = X \times Y$, we have the result. [1.2 - A]

1.4 Elementary Remarks.

1. Let $X = \prod_{i \in I} X_i$ and $\pi_{I'}: X \to \prod_{i \in I'} X_i$ where $I' \subset I$. Then we know that $\pi_{I'}$ is continuous in any product topology and hence if X is weakly m-n compact, then every sub-product of X is weakly m-n compact.

2. Let X be a paracompact T_2- space. Then every open cover of X can be refined by a locally finite open cover such that closures of the memembers are contained in the members of the

original cover. Hence weakly m-n compact, paracompact T_2-space is m-n compact, provided the space has cardinality less than or equal to m.

3. Let X be a space and let A be a weakly m-n compact subset of X. Then $\overline{A}$ is weakly m-n compact.

4. Let $f:X \to Y$ be continuous and suppose X is weakly m-n compact. Then we know that f(X) is weakly m-n compact and hence $c\ell_Y f(X)$. There is a simple question available. Under the hypothesis of 4, is it true that every subspace between f(X) and $c\ell_Y f(X)$ is weakly m-n compact? Answer is yes.

General Remark

Topologically descriptive good notion of convergence is available in terms of filters. We also know that topological concepts can be described conveniently in terms of filters and in particular since compactness-like properties can be described using intersection properties of families of sets, filters play an important role in the theory of compact spaces.

Filters $\longrightarrow$ compact spaces,

Open Filters $\longrightarrow$ H(i) spaces,

Z-Filters $\longrightarrow$ Stone-Čech compactification (Closed Filters-Wallman compactification),

m-n Filters $\longrightarrow$ m-n compact spaces

The idea of this object can be found in much earlier work but actual usage dates back to 1930-1940 (H. Cartan-Paris).